Lectures on infinite interacting systems

LECTURES IN MATHEMATICS

Department of Mathematics
KYOTO UNIVERSITY

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LECTURES ON INFINITE INTERACTING SYSTEMS

BY

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Department of Mathematics
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CONTENTS

Preface

These notes are based on lectures given by me at R.I.M.S. during May and June of 1977. The topic discussed is the theory of stochastic Ising models and the application of this theory to the study of Gibbs states. All the material presented has appeared elsewhere in various papers by R. Holley, T. Liggett, myself and others. My purpose has been to give a survey of the kinds of problems that one encounters in this relatively new area of probability theory. I have also tried to provide an introduction to some of the techniques which have proved successful in attacking these problems. I do not claim that my treatment of this subject has been exhaustive in any direction; indeed, an exhaustive treatment would be foolish at this time since it is still too early to predict what directions the field will take. My choice of topics has been guided exclusively by my own prejudices about what is interesting. I will consider these notes a success if their inadequacy provokes someone into filling the huge gaps which makes them inadequate.

I would like to thank K. Itô for inviting me to Kyoto and giving me the opportunity to present this material both in my lectures as well as in these notes. Also I am grateful to Kotani, Kasahara, Higuchi, and Asano for transforming my rough notes into the form in which they appear here.

November 1977 D.W. Stroock

I. Background:

The origin of the interest in this subject lies in the following formulation of equilibrium statistical mechanics for Ising type models.

(1) Notation : We define,

$E = (\{-1, 1\})^{Z^d}$ with the product topology,

$\mathcal{B} = \mathcal{B}_E$, the Borel field over E.

Elements of E are denoted by η $(= \{\eta_k : k \in Z^d\})$. Given $\Lambda \subseteq Z^d$, let

$$E_\Lambda = \{-1, 1\}^\Lambda$$

and let $\mathcal{B}^\Lambda(\tilde{\mathcal{B}}^\Lambda)$ be the inverse image of $\mathcal{B}_{E_\Lambda}$ $(\mathcal{B}_{E_{\Lambda^c}})$ under the natural projection map of E onto E_Λ (E_{Λ^c}). Given Λ_1 and Λ_2 such that $\Lambda_1 \cap \Lambda_2 = \emptyset$, for $\alpha \in E_{\Lambda_1}$ and $\beta \in E_{\Lambda_2}$, define $\alpha \times \beta \in E_{\Lambda_1 \cup \Lambda_2}$ by

$$(\alpha \times \beta)_k = \begin{cases} \alpha_k & \text{if } k \in \Lambda_1 \\ \beta_k & \text{if } k \in \Lambda_2. \end{cases}$$

Finally, if $\Lambda \subseteq Z^d$, let $|\Lambda| = \text{card}\ (\Lambda)$ and define

$$\hat{E} = \{F \subseteq Z^d : |F| < \infty\}.$$

If $F \in \hat{E}$, then $\chi_F : E \to \{-1, 1\}$ is the function given by

$$\chi_F(\eta) = \prod_{k \in F} \eta_k, \qquad \eta \in E.$$

(2) Definition : A potential is a function $F \to J_F$ on $\hat{E}\backslash\{\emptyset\}$ into R^1 such that :

$$\sup_{k \in Z^d} \sum_{F \ni k} |J_F| < \infty. \tag{3}$$

Given a potential $\{J_F : F \in \hat{E}\backslash\{\emptyset\}\}$ and a set $\Lambda \in \hat{E}\backslash\{\emptyset\}$, we define the conditional energy of the state $\alpha \in E_\Lambda$ given the state $\beta \in E_{\Lambda^c}$ by

(4) $$U_\Lambda(\alpha ; \beta) = \sum_{F \cap \Lambda \neq \emptyset} J_F \chi_F(\alpha \times \beta).$$

Also, for each $\beta \in E_{\Lambda^c}$, define the probability measure $\mu_\Lambda(\cdot ; \beta)$ on $(E_\Lambda, \mathcal{B}_{E_\Lambda})$ by

(5) $$\mu_\Lambda(\{\alpha\} ; \beta) = \exp(- U_\Lambda(\alpha, \beta))/Z_\Lambda(\beta), \ \alpha \in E_\Lambda$$

where

(6) $$Z_\Lambda(\beta) = \sum_{\alpha \in E_\Lambda} \exp(- U_\Lambda(\alpha ; \beta)).$$

(7) Remark ; Assuming that $\frac{1}{kT} = 1$, the probability measure $\mu_\Lambda(\cdot ; \beta)$ is exactly the one prescribed by Gibbs to describe the equilibrium distribution of the states $\alpha \in E_\Lambda$ when the energy of state α is given by (4). Since we want to interpret $U_\Lambda(\alpha ; \beta)$ as the conditional energy of state $\alpha \in E_\Lambda$ given state $\beta \in E_{\Lambda^c}$, we would like to think of $\mu_\Lambda(\{\alpha\} : \beta)$ as the conditional probability of $\{\eta : \eta|_\Lambda = \alpha\}$ given that $\eta|_{\Lambda^c} = \beta$. That it is consistent to do so is the content of the next statement (cf. Dobrushin, Fnal. Anal. Appl. 2, (1968)).

(8) Proposition : If $\emptyset \neq \Lambda_1 \subseteq \Lambda_2 \in \hat{E}$, $\alpha \in E_{\Lambda_1}$, $\beta \in E_{\Lambda_2 \setminus \Lambda_1}$, and $\gamma \in E_{\Lambda_2^c}$, then

$$\mu_{\Lambda_1}(\{\alpha\} : \beta \times \gamma)$$
$$= \mu_{\Lambda_2}(\{\sigma \in E_{\Lambda_2} : \sigma|_{\Lambda_1} = \alpha\} | \{\sigma \in E_{\Lambda_2} : \sigma|_{\Lambda_2 \setminus \Lambda_1} = \beta\} ; \gamma)$$

Thus, there exists a probability measure μ on $(E, \mathcal{B})$ with the property that for every $\Lambda \in \hat{E} \setminus \{\emptyset\}$ the map $\beta \to \mu_\Lambda(\cdot ; \beta)$ is a regular conditional probability distribution of μ given $\tilde{\mathcal{B}}^{\Lambda}$ (abbreviated by : r. c. p. d. of $\mu|_{\tilde{\mathcal{B}}^{\Lambda}}$).

<u>(9) Definition</u> : Given $J = \{J_F : F \in \hat{E}\backslash\{\emptyset\}\}$, let $G(J)$ be the set of all probability measures μ on $(E, \mathcal{B})$ such that, for each $\Lambda \in \hat{E}\backslash\{\emptyset\}$, $\beta \to \mu_{\Lambda}(\cdot\ ;\ \beta)$ is an r. c. p. d. of $\mu|_{\tilde{\mathcal{B}}^{\Lambda}}$. An element of $G(J)$ is called a Gibbs state for potential J.

<u>(10) Proposition</u> : $G(J)$ is a non-empty compact convex set and $\mu \in G(J)$ if and only if, for each $k \in Z^d$, $\beta \to \mu_{\{k\}}(\cdot\ ;\ \beta)$ is an r. c. p. d. of $\mu|_{\tilde{\mathcal{B}}^{\{k\}}}$.

<u>(11) Remark</u> : The central problem in this field is to study the set $G(J)$. In particular, one wants to know when $G(J)$ has more than one element. There are many fascinating aspects of this problem. The tack that we are going to talk is only one of many, and by means the most successful.

II. Glauber Type Models :

In section I, we introduced the notion of a Gibbs state for a potential $J = \{J_F : F \in \hat{E}\backslash\{\emptyset\}\}$. However, in spite of the fact that a Gibbs state should be the equilibrium state of some sort of dynamical system, no dynamics has been mentioned yet. What we are about to do is remedy this situation. The idea is due to Glauber.

Consider a stochastic process $\eta(t)$, with state space E, having the following infinitesimal characteristics :

$$(1) \quad P(\eta_k(t+h) \neq \eta_k(t)\ |\eta(s),\ \ 0 \le s \le t) = h\, c_k(\eta(t)) + o(h), \quad k \in Z^d,$$

$$(2) \quad P(\eta_k(t+h) \neq \eta_k(t) \text{ and } \eta_\ell(t+h) \neq \eta_\ell(t) \;|\eta(s), \quad 0 \le s \le t) = o(h), \qquad k \neq \ell \in Z^d,$$

where $c : E \to ([0, \infty))^{Z^d}$ is a continuous function satisfying the following detailed balance condition :

$$(3) \quad c_k(\eta) \exp(-U_{\{k\}}(\eta_k ; \tilde{\eta}^{\{k\}})) = c_k({}^k\eta) \exp(-U_{\{k\}}(-\eta_k ; \tilde{\eta}^{\{k\}})),$$

where ${}^k\eta$ is the element of E given by

$$(4) \quad ({}^k\eta)_\ell = \begin{cases} -\eta_k & \text{if} \quad \ell = k \\ \eta_\ell & \text{if} \quad \ell \neq k, \end{cases}$$

and $\tilde{\eta}^{\{k\}}$ denotes $\eta|_{Z^d \setminus \{k\}}$. An easy formal computation leads to the conclusion that any element $\mu \in G(J)$ is a stationary (in fact, reversible) measure for $\eta(t)$. Rather than carry out this computation, we will consider a trivial example.

III. The Case of No Interaction :

Suppose that

$$J_F = \begin{cases} a & \text{if} \quad |F| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$U_{\{k\}}(\pm 1 \; ; \beta) = \pm a, \quad k \in Z^d \quad \text{and} \quad \beta \in E_{Z^d \setminus \{k\}}.$$

Thus the coefficients

$$c_k(\eta) = e^{a\eta_k}, \quad k \in Z^d \quad \text{and} \quad \eta \in E,$$

satisfy (3) of section II. Clearly $G(J)$ in this case consists of exactly one element μ, namely :

$$(1) \qquad \mu = \left(\frac{e^{-a}\delta_{\{1\}}(\cdot) + e^{a}\delta_{\{-1\}}(\cdot)}{e^{-a} + e^{a}}\right)^{Z^d}.$$

The Glauber type process in this case has independent coordinates and each coordinate has transition probability function (abbreviated by : tr. pr. fn.) :

$$P_k(t, \pm 1, \cdot) = \frac{(e^{-a} \pm e^{\pm a - bt})\delta_{\{1\}}(\cdot) + (e^{a} \mp e^{\pm a - bt})\delta_{\{-1\}}(\cdot)}{b}$$

where $b = e^{-a} + e^{a}$. Thus the transition probability function for the Glauber type process is

$$(2) \qquad P(t, \eta, \cdot) = \prod_{k \in Z^d} P_k(t, \eta_k, \cdot).$$

It is easy to check that the measure μ in (1) is a reversible measure for $P(t, \eta, \cdot)$. Indeed, one need only do so one coordinate at a time.

In spite of the simplicity of the process in this case, it already indicates some of the pathological characteristics of this sort of stochastic process. We list some of these below :

i) $P(s, \eta, \cdot) \perp P(t, \eta, \cdot)$ if $0 \leq s < t$,

ii) $\perp$ $P(t, \eta, \cdot)$ $P(t, \eta', \cdot)$ if $\sum_k |\eta_k - \eta'_k| = \infty$,

iii) $P(t, \eta, \cdot) \to \mu$, weakly as $t \to \infty$, but $P(t, \eta, \cdot) \perp \mu$,

iv) there is no reference measure,

V) $P(t, \eta, \cdot)$ is Feller, but not strongly Feller continuous.

IV. A Little Functional Analysis.

In this section we will make the rigorous connection between Gibbs states for a given potential and Glauber type process satisfying the detailed balance condition (3) of section II.

For each $k \in Z^d$ and $\phi \in B(E)$, define

$$(1) \qquad \Delta_k\phi(\eta) = \phi_{,k}(\eta) = \phi({}^k\eta) - \phi(\eta), \qquad \eta \in E.$$

We denote by $\mathcal{D}$ the set of $\phi \in C(E)$ such that $\phi_{,k} \equiv 0$ for all but a finite number of k's. Given a continuous function $c : E \to ([0, \infty))^{Z^d}$, define the operator

$$(2) \qquad L = \sum_k c_k\Delta_k.$$

on $\mathcal{D}$. Throughout this section we will be assuming that there is a unique Feller semi-group $\{T_t : t > 0\}$ on $C(E)$ such that

$$(3) \qquad T_t\phi - \phi = \int_0^t T_s L\phi \quad ds, \quad \phi \in \mathcal{D}.$$

In addition, we will be assuming that if

$$L^{(n)} = \sum_{|k|\le n} c_k\Delta_k,$$

then

$$(4) \qquad \|e^{tL^{(n)}}\phi - T_t\phi\| \to 0 \quad \text{as} \quad n \to \infty, \qquad \phi \in C(E).$$

(Here, and throughout, $|k| = \max_{1\le j\le d} |k_j|$ and $\|\phi\| = \sup_\eta |\phi(\eta)|$.)

Note that $L^{(n)}$ is a bounded operator and so $e^{tL^{(n)}}$ is trivially defined. Moreover, it is easily checked that if

$$T_t^{(n)} = e^{tL^{(n)}}, \quad t > 0,$$

then $\{T_t^{(n)} : t > 0\}$ is a Feller semi-group on $C(E)$.

(5) Definition : Let $\mathcal{G}(L)$ denote the set of probability measures μ on $(E, \mathcal{B})$ such that

$$(6) \qquad \int \phi L\psi \, d\mu = \int \psi L\phi \, d\mu, \qquad \phi, \psi \in \mathcal{D}.$$

(7) Theorem : Given a probability measure μ on $(E, \mathcal{B})$, let $\mu_{\{k\}}(\cdot|\tilde{\eta}^{\{k\}})$ denote an r. c. p. d. of $\mu|_{\tilde{\mathcal{B}}^{\{k\}}}$. Then the following are equivalent :

i) $\mu \in \mathcal{G}(L)$,

ii) for each $k \in Z^d$,

$$\mu_{\{k\}}(\{\eta_k\}|\tilde{\eta}^{\{k\}})c_k(\eta) = \mu_{\{k\}}(\{-\eta_k\}|\tilde{\eta}^{\{k\}})c_k({}^k\eta). \quad (\text{a.s. } \mu),$$

iii) $\int \phi L\psi \, d\mu = -\frac{1}{2}\sum_k \int c_k \phi,_k \psi,_k \, d\mu, \quad \phi, \psi \in \mathcal{D}$,

iv) $\int \phi T_t\psi \, d\mu = \int \psi T_t\phi \, d\mu, \quad \phi, \psi \in C(E)$.

In particular, if the c_k's satisfy condition (3) in II relative to some potential J, then $\mathcal{G}(J) \subseteq \mathcal{G}(L)$, and if, in addition, $c_k > 0$ for each $k \in Z^d$, then $\mathcal{G}(J) = \mathcal{G}(L)$.

Proof : We first prove that i) $\Rightarrow$ ii) $\Rightarrow$ iii) $\Rightarrow$ i).

i) $\Rightarrow$ ii) : Let $\phi(\eta) = \eta_k$ and choose $\psi \in \mathcal{D}$ such that $\psi,_k \equiv 0$. Then,

$$L(\phi \cdot \psi) = \phi L\psi + \psi L\phi.$$

Thus, if $\mu \in \mathcal{G}(L)$, then $\int \psi L\phi \, d\mu = 0$. But this means that :

$$\int \psi(\tilde{\eta}^{\{k\}}) c_k(1 \times \tilde{\eta}^{\{k\}}) \mu_{\{k\}}(\{1\}|\tilde{\eta}^{\{k\}})\mu(d\eta)$$
$$= \int \psi(\tilde{\eta}^{\{k\}}) c_k(-1 \times \tilde{\eta}^{\{k\}}) \mu_{\{k\}}(\{-1\}|\tilde{\eta}^{\{k\}})\mu(d\eta)$$

for any $\psi \in \mathcal{D}$ such that $\psi,_k \equiv 0$.

ii) $\Rightarrow$ iii) : Let $\phi, \psi \in \mathcal{D}$. Then if μ satisfies ii),

$$\int \phi L\psi \, d\mu = \sum_k \int c_k \phi \psi,_k \, d\mu$$
$$= \sum_k \int c_k(1 \times \tilde{\eta}^{\{k\}}) \phi(1 \times \tilde{\eta}^{\{k\}}) \psi,_k(1 \times \tilde{\eta}^{\{k\}}) \mu_{\{k\}}(\{1\}|\tilde{\eta}^{\{k\}})\mu(d\eta)$$
$$+ \sum_k \int c_k(-1 \times \tilde{\eta}^{\{k\}}) \phi(-1 \times \tilde{\eta}^{\{k\}}) \psi,_k(-1 \times \tilde{\eta}^{\{k\}}) \mu_{\{k\}}(\{-1\}|\tilde{\eta}^{\{k\}})\mu(d\eta$$
$$= -\sum_k \int c_k(1 \times \tilde{\eta}^{\{k\}}) \phi,_k(1 \times \tilde{\eta}^{\{k\}}) \psi,_k(1 \times \tilde{\eta}^{\{k\}}) \mu_{\{k\}}(\{-1\}|\tilde{\eta}^{\{k\}})\mu(d\eta)$$
$$= -\frac{1}{2} \sum_k \int c_k(\eta) \phi,_k(\eta) \psi,_k(\eta) \mu(d\eta).$$

iii) $\Rightarrow$ i) is obvious.

i) $\Leftrightarrow$ iv) : Clearly iv) $\Rightarrow$ i). To prove i) $\Rightarrow$ iv), define

$$c_k^{(n)} = \begin{cases} c_k & \text{if} \quad |k| \le n \\ 0 & \text{if} \quad |k| > n. \end{cases}$$

Then $L^{(n)} = \sum_k c_k^{(n)} \Delta_k$. Since if $\mu \in G(L)$ then μ satisfies ii) for the c_k's, it is clear that $\mu \in G(L)$ implies μ satisfies ii) for the $c_k^{(n)}$'s. But ii) $\Rightarrow$ i), and so $\mu \in G(L) \Rightarrow \mu \in G(L^{(n)})$. On the other hand, $L^{(n)}$ is bounded and $e^{tL^{(n)}}$ is given by a power series. Thus

$$\mu \in G(L^{(n)}) \Rightarrow \int \phi T_t^{(n)} \psi \, d\mu = \int \psi T_t^{(n)} \phi \, d\mu, \quad \phi, \psi \in \mathcal{D}.$$

Because of (4), it is now clear that i) $\Rightarrow$ iv).

The relationship between $G(J)$ and $G(L)$ is obvious from ii) together with (10) of section I.

<u>(8) Theorem</u> : For each $\mu \in G(L)$ there is a unique extention of $\{T_t : t > 0\}$ to $L^2(\mu)$ as a semi-group of self-adjoint contractions $\{T_t^\mu : t > 0\}$. Moreover, if $\{E_\lambda^\mu : \lambda > 0\}$ is the resolution of the identity in $L^2(\mu)$ such that

$$T_t^\mu = \int_0^\infty e^{-\lambda t}\, dE_\lambda^\mu, \quad t > 0,$$

then :

$$(9) \quad \int \lambda d(E_\lambda^\mu \phi, \phi) = \frac{1}{2} \sum_k \int c_k |\phi,_k|^2 d\mu, \quad \phi \in L^2(\mu).$$

Thus, if $\sum_k \int c_k(|\phi,_k|^2 + |\psi,_k|^2)d\mu < \infty$ for some pair $\phi, \psi \in L^2(\mu)$, then

$$(10) \quad \int \lambda d(E_\lambda^\mu \phi, \psi) = \frac{1}{2} \sum_k \int c_k \phi,_k \overline{\psi},_k \, d\mu.$$

<u>Proof</u> : In view of iv) in the preceding theorem plus the observation that :

$$\int |T_t\phi|^2 \, d\mu \leq \int T_t(|\phi|^2)d\mu = \int |\phi|^2 \, d\mu,$$

we need only prove (9).

Clearly, for each $n \geq 1$ the relation

$$\int_0^\infty \lambda d(E_\lambda^{(n)}\phi, \phi) = - \int \phi L^{(n)}\phi \, d\mu$$
$$= \frac{1}{2} \sum_{|k|\leq n} \int c_k |\phi,_k|^2 \, d\mu, \qquad \phi \in \mathcal{D},$$

extends to $\phi \in L^2(\mu)$. (Here we have used the fact that $\mu \in G(L) \Rightarrow$

$\mu \in G(L^{(n)})$ and $\{E_\lambda^{(n)} : \lambda > 0\}$ is defined for $\{T_t^{(n)} : t > 0\}$ in the same way as $\{E_\lambda^\mu : \lambda > 0\}$ is for L.) In particular, if $t > 0$, then

$$\frac{1}{2}\sum_k \int c_k |\phi,_k|^2 \, d\mu \geq \int_0^\infty \lambda e^{-2\lambda t} \, d(E_\lambda^{(n)}\phi, \phi)$$

$$= \frac{1}{2}\sum_k \int c_k |T_t^{(n)}\phi|^2 \, d\mu.$$

But for $\phi \in L^2(\mu)$ and $t > 0$, we have for any $\psi \in \mathcal{D}$:

$$\left|\int_0^\infty \lambda e^{-2\lambda t} \, d(E_\lambda^{(n)}\phi, \phi) - \int_0^\infty \lambda e^{-2\lambda t} \, d(E_\lambda^\mu\phi, \phi)\right|$$

$$\leq \left|\int_0^\infty (1 - 2t)e^{-2\lambda t}[(E_\lambda^{(n)}\phi, \phi) - (E_\lambda^\mu\psi, \psi)] \, d\lambda\right|$$

$$+ \left|(T_{2t}^{(n)}\psi, L^{(n)}\psi) - (T_{2t}\psi, L\psi)\right|$$

$$+ \left|\int_0^\infty (1 - 2t)e^{-2\lambda t}[(E_\lambda^\mu\phi, \phi) - (E_\lambda^\mu\psi, \psi)] \, d\lambda\right| .$$

Since $\mathcal{D}$ is dense in $L^2(\mu)$, it follows that

$$\int_0^\infty \lambda e^{-2\lambda t} \, d(E_\lambda^{(n)}\phi, \phi) \to \int_0^\infty \lambda e^{-2\lambda t} \, d(E_\lambda^\mu\phi, \phi)$$

as $n \to \infty$. We therefore have :

$$\frac{1}{2}\sum_k \int c_k \mid \phi,_k \mid^2 \, d\mu \geq \int_0^\infty \lambda e^{-2\lambda t} \, d(E_\lambda^\mu\phi, \phi)$$

$$\geq \frac{1}{2}\sum_k \int c_k \mid T_t^\mu\phi \mid^2 \, d\mu$$

for all $\phi \in L^2(\mu)$ and $t > 0$. Letting $t \downarrow 0$, we get (9).

<u>(11) Corollary</u> : For each $\mu \in G(L)$ the null space $N(L^\mu)$ of the generator L^μ of $\{T_t : t > 0\}$ coincides with the set of $\phi \in L^2(\mu)$ such that

$$(12)\qquad \sum_k \int c_k \mid \phi,_k \mid^2 d\mu = 0.$$

Moreover, if

$$(13)\qquad \alpha_0 \equiv \inf \{\sum_k \int c_k \mid \phi,_k \mid^2 d\mu/2\|\phi - E_0^\mu\phi\|^2_{L^2(\mu)} : \phi \neq E_0^\mu\phi\}$$

then

$$(14)\qquad E_\lambda^\mu - E_0^\mu = 0 \quad \text{for} \quad 0 \le \lambda < \alpha_0,$$

and α_0 is the largest number α such that

$$(15)\qquad \|T_t^\mu\phi - E_\lambda^\mu\|_{L^2(\mu)} \le e^{-\alpha t}\|\phi\|_{L^2(\mu)}, \quad t > 0 \text{ and } \phi \in L^2(\mu)$$

<u>(16) Lemma</u> : Let $\mu \in G(L)$. If $c_k > 0$ for all $k \in Z^d$, then

$$(17)\qquad N(L^\mu) = \{\phi \in L^2(\mu) : \phi \text{ is } T\text{-measurable.} ,$$

where $T = \bigcap_{n=1}^{\infty} \tilde{B}^{\Lambda_n}$ with $\Lambda_n = \{k \in Z^d : |k| \le n\}$ (i.e. T is the tail field.).

<u>Proof</u> : From (12) and the positivity of the c_k's, we see that $\phi \in L^2(\mu)$ is in $N(L^\mu)$ if and only if $\phi \in \Phi_0$, where Φ_0 is the set of all B-measurable ϕ such that $\phi,_k = 0$ (a.s. μ) for all $k \in Z^d$. In particular, if $\phi \in L^2(\mu)$ is T-measurable, then certainly $\phi \in N(L^\mu)$. To prove the opposite inclusion, we need only show that $\phi \in \Phi_0 \Rightarrow \phi$ is T-measurable. That is, we must show that if $\phi \in \Phi_0$, then for all $\Lambda \in E \setminus \{\emptyset\}$

$$\phi = \phi \circ \Pi_\Lambda \qquad (\text{a.s. } \mu),$$

where $\Pi_\Lambda : E \to E$ is given by

$$(\Pi_\Lambda\ \eta)_k = \begin{cases} 1 & \text{if } k \in \Lambda \\ \eta & \text{if } k \notin \Lambda. \end{cases}$$

To this end, we first observe that $\phi \in \Phi_0 \Rightarrow \phi \circ \Pi_\Lambda \in \Phi_0$ for all $\Lambda \in \hat{E} \setminus \{\emptyset\}$ Indeed, from $c_k > 0$, $k \in Z^d$, and ii) of theorem (7), we see that $\mu \circ \Pi_\Lambda^{-1} << \mu$. Thus, since

$$(\phi \circ \Pi_\Lambda)_{,k} = \begin{cases} 0 & \text{if } k \in \Lambda \\ \phi_{,k} \circ \Pi_\Lambda & \text{if } k \notin \Lambda, \end{cases}$$

it follows that $\phi \in \Phi_0 \Rightarrow \phi \circ \Pi_\Lambda \in \Phi_0$. Working by induction, we now see that it is only necessary to check that $\phi \in \Phi_0 \Rightarrow \phi = \phi \circ \Pi_{\{k\}}$ (a.s. μ) for each $k \in Z^d$. But,

$$\phi \circ \Pi_{\{k\}} = \phi + \frac{1 - \eta_k}{2} \phi_{,k} ,$$

and so there is nothing more to do.

<u>(18)</u> <u>Lemma</u> : Let $\mu \in G(L)$ and suppose that ν is a probability measure on $(E, \mathcal{B})$ such that $\nu << \mu$ and $\frac{d\nu}{d\mu}$ is $\mathcal{T}$-measurable. Then $\nu \in G(L)$.

<u>Proof</u> : Let $f = \frac{d\nu}{d\mu}$. Given $\phi, \psi \in \mathcal{D}$, note that $(f \cdot \phi)_{,k} = f \cdot \phi_{,k}$ (a.s. μ), and so

$$\sum_k \int c_k((f\phi)_{,k}^2 + \psi_{,k}^2)\ d\mu < \infty.$$

Thus

$$
\begin{aligned}
- \int \phi \mathcal{L}\psi \, d\nu &= - \int (f \cdot \phi) \mathcal{L}^{\mu}\psi \, d\mu \\
&= \int \lambda \, dE^{\mu}_{\lambda}(f \cdot \phi, \phi) \\
&= \frac{1}{2} \sum_{k} \int c_k (f \cdot \phi)_{,k} \, \psi_{,k} \, d\mu \\
&= \frac{1}{2} \sum_{k} \int c_k \, \phi_{,k} \, \psi_{,k} \, d\nu
\end{aligned}
$$

By theorem (7), this proves that $\nu \in \mathcal{G}(\mathcal{L})$.

(19) Notation : Let $\{P_\eta : \eta \in E\}$ be the Markov family of probability measures on path space $(\Omega, \mathcal{M})$ determined by the semi-group $\{T_t : t > 0\}$. For $t > 0$, let $\theta_t : \Omega \to \Omega$ denote the time-shift operator. Finally, if μ is a probability measure on $(E, \mathcal{B})$, let

$$P_\mu = \int P_\eta \, \mu(d\eta).$$

(20) Lemma : If $\mu \in \mathcal{G}(\mathcal{L})$ and $\mathcal{N}(\mathcal{L}^\mu) = \{\phi \in L^2(\mu) : \phi = \int \phi \, d\mu \text{ (a.s. } \mu)\}$, then the dynamical system $(\Omega, \mathcal{M}, \theta_t, P_\mu)$ is ergodic.

Proof : This is a standard argument for Markov processes.

The next theorem summarizes the last few results.

(21) Theorem : Assume that $c_k > 0$ for all $k \in Z^d$. If $\mu \in \mathcal{G}(\mathcal{L})$, then $T^\mu_t \to E^\mu[\cdot \mid \mathcal{T}]$ strongly in $L^2(\mu)$. Moreover, the following are equivalent :

i) μ is an extreme point of $\mathcal{G}(\mathcal{L})$,

ii) $\mathcal{T}$ is μ-trivial,

iii) $E^\mu_0 \, \phi = \int \phi \, d\mu$ (a.s. μ), $\phi \in L^2(\mu)$,

iv) $T^\mu_t \, \phi \to \int \phi \, d\mu$ in $L^2(\mu)$, $\phi \in L^2(\mu)$,

v) $(\Omega, \mathcal{M}, \theta_t, P_\mu)$ is ergodic.

(22) Remark : Let $J = \{J_F : F \in \hat{E} \setminus \{\emptyset\}\}$ be a potential. One of the things that we have just seen is that $G(J) = G(L)$ whenever $L = \sum_k c_k \Delta_k$ with the c_k's positive and satisfying (3) of section II. One such choice of c_k's is :

$$\text{(23)} \qquad c_k^0(\eta) = [1 + \exp(2 \sum_{F \ni k} J_F \chi_F(\eta))], \quad k \in Z^d.$$

However, there are many others. In fact, if we choose any continuous $b : E \to ([0, \infty))^{Z^d}$ so that $b_k(\eta) = b_k({}^k\eta)$ for all $k \in Z^d$ and $\eta \in E$, then the functions

$$c_k(\eta) = b_k(\eta)\, c_k^0(\eta), \quad k \in Z^d,$$

also satisfies the detailed balance condition. Unfortunately, there is no really "canonical" choice of the c_k's for a given J. It is therefore of some interest to know what properties all the choices share in common. Among the few results in this direction are those which can be read off from equation (9) about the spectral properties of L^μ. See H and S, Z. Wahr, 35, (1976) for a further discussion of this and related matters. Also, see section XII below.

V. The Martingale Problem for L :

We have not as yet discussed how one constructs Markov processes having given rates c_k. We will do so now.

Let $\Omega = D([0, \infty), E)$ be the space of right continuous functions ω on $[0, \infty) \to E$ having left limits and endow Ω with the usual Skorohod topology. Given $\omega \in \Omega$, let $\eta(t, \omega)$ denote the position of ω at time t. Let $M = \mathcal{B}_\Omega$ and for $t \geq 0$ set $M_t = \sigma(\eta(s) ; 0 \leq s \leq t)$.

Given a continuous function $c : E \to ([0, \infty))^{Z^d}$, define $L = \sum_k c_k \Delta_k$ on $\mathcal{D}$ accordingly. We will say that the probability measure P on $(\Omega, \mathcal{M})$ <u>solves the martingale problem for L starting from η</u> (abbr. $P \sim L$ starting from η) if :

(1) $\quad P(\eta(0) = \eta) = 1,$

and

(2) $\quad (\phi(\eta(t)) - \int_0^t L\phi(\eta(s))ds, \mathcal{M}_t, P)$ is a martingale for all $\phi \in \mathcal{D}$.

The following theorem is easily derived using well-known techniques from the calculus of martingales.

<u>(3) Theorem</u> : Let $\mathcal{D}([0, T])$ be the set of continuous functions $\phi : [0, T] \times E \to R^1$ such that $\frac{\partial \phi}{\partial t}$ exists and is continuous on $[0, T] \times E$ and $\phi,_k \equiv 0$ for all but a finite number of k's. If $P \sim L$ starting at η, then for all $\phi \in \mathcal{D}([0, T])$:

$$(\phi(t \wedge T, \eta(t \wedge T)) - \int_0^{t \wedge T} (\frac{\partial \phi}{\partial s} + L\phi)(s, \eta(s))ds, \mathcal{M}_t, P)$$

is a martingale. Also, if $\tau : \Omega \to [0, \infty)$ is a stopping time and $\omega \to P^\tau_\omega$ is an r. c. p. d. of $P|\mathcal{M}_\tau$, then there is a P-null set $N \in \mathcal{M}_\tau$ such that for all $\omega \notin N$ and all $\phi \in \mathcal{D}$

$$(\phi(\eta(t)) - \int_{\tau(\omega)}^t L\phi(\eta(s))ds, \mathcal{M}_t, P^\tau_\omega)$$

is a martingale after time $\tau(\omega)$. In particular, for all $\omega \notin N$, $P^\tau_\omega \circ \theta_{\tau(\omega)}^{-1} \sim L$ starting at $\eta(\tau(\omega), \omega)$, where $\theta_t : \Omega \to \Omega$ is the usual time-shift operator (i.e. $\eta(\cdot, \theta_t \omega) = \eta(\cdot + t, \omega)$).

(4) Remark : We want to check that if $P \sim L$ starting from η, then P has the infinitesimal characteristics of a Glauber-type process with rates c_k. To this end, let $\phi(\eta) = \frac{1}{2}\eta_k$. Then, since

$$(\phi(\eta(t)) + \int_0^t \eta_k(s)c_k(\eta(s))ds,\ M_t,\ P)$$

is a martingale, we see that

$$\begin{aligned} &P(\eta_k(t) \neq \eta_k(s) \quad \text{for some} \quad t \in [s, s+h] \mid M_s) \\ &\quad = \left| E^P[\phi(\eta((s+h) \wedge \tau) - \phi(\eta(s)) \mid M_s] \right| \\ &\quad = E^P[\int_s^{(s+h)\wedge\tau} c_k(\eta(u))du \mid M_s] \\ &\quad = hc_k(\eta(s)) + o(h), \end{aligned}$$

where $\tau = \inf\{t \geq s : \eta_k(t) \neq \eta_k(s)\}$. Next, let $\omega \to P_\omega^s$ be an r. c. p. d. of $P|M_s$. Given $k \neq \ell$, set

$$\phi_\omega(\eta) = \frac{1}{4}(\eta_k - \eta_k(s, \omega))(\eta_\ell - \eta_\ell(s, \omega))$$

and

$$\tau = \inf\{t \geq s : \eta_k(t) \neq \eta_k(s) \quad \text{and} \quad \eta_\ell(t) \neq \eta_\ell(s)\}.$$

Then

$$\begin{aligned} &P(\eta_k(t) \neq \eta_k(s) \quad \text{and} \quad \eta_\ell(t) \neq \eta_\ell(s) \quad \text{for some} \quad t \in [s, s+h] \mid M_s) \\ &\quad = \left| E[\phi_\omega(\eta((s+h) \wedge \tau)) \mid M_s] \right| \\ &\quad \leq \frac{1}{2} \left| E^{P_\omega^s}[\int_s^{(s+h)\wedge\tau} (c_k(\eta(u))(\eta_k(u) - \eta_k(s, \omega)) \right. \\ &\qquad\qquad \left. + c\ (\eta(u))(\eta_\ell(u) - \eta_\ell(u, \omega)))du] \right| \\ &\quad = O(h^2). \end{aligned}$$

(5) Example : Suppose that

$$c_k(\eta) = a_k(1 + \alpha_k\eta_k), \qquad k \in Z^d,$$

where $a_k \geq 0$ and $|\alpha_k| \leq 1$. Define the transition probability function $P_k(t, \eta_k, \cdot)$ on $\{-1, 1\}$ by :

$$P_k(t, \pm 1, \cdot) = \left(\frac{1-\alpha_k}{2} + \frac{\alpha_k \pm 1}{2} e^{-2a_k t}\right)\delta_{\{1\}}(\cdot) + \left(\frac{1+\alpha_k}{2} - \frac{\alpha_k \pm 1}{2} e^{-2a_k t}\right)\delta_{\{-1\}}(\cdot)$$

and set

$$P(t, \eta, \cdot) = \prod_k P_k(t, \eta_k, \cdot).$$

For $\phi \in \mathcal{D}$, note that

$$u(t, \eta) = \int \phi(\beta)P(T - t, \eta, d\beta), \qquad 0 \leq t \leq T,$$

is in $\mathcal{D}([0, T])$ and that

$$\frac{\partial u}{\partial t} + Lu = 0, \qquad 0 \leq t < T,$$

$$\lim_{t \uparrow T} u(t, \eta) = \phi(\eta).$$

Thus, if $P \sim L$ starting from η, then

$$(u(t \wedge T, \eta(t \wedge T)), M_t, P)$$

is a martingale, and so if $0 \leq t < T$:

$$E^P[\phi(\eta(T)) \mid M_t] = u(t, \eta(t))$$
$$= \int \phi(\beta)P(T - t, \eta(t), d\beta) \qquad \text{(a.s. } P).$$

Combining this with the fact that $P(\eta(0) = \eta) = 1$, we see that P must be the homogeneous Markov process with transition probability function $P(t, \eta, \cdot)$ conditioned to start at η. This example provides further evidence that solutions to the martingale problem are the processes for which we are looking.

VI. Existence and Elementary Consequences of Uniqueness :

In this section we prove that solutions to the martingale problem exist for very general coefficients c. We also point out that under very general conditions one can use these solutions to construct a Markov semi-group $\{T_t : t > 0\}$ on $B(E)$ such that

$$T_t\phi - \phi = \int_0^t T_s \mathcal{L}\phi \, ds, \qquad t > 0 \text{ and } \phi \in \mathcal{D}.$$

However, no assertions about the uniqueness of any of these quantities will be proved here; that will be postponed until the next few sections. Instead, what we will do here is point out some of the consequences of uniqueness, with the idea of motivating interest in the problem of finding conditions under which uniqueness must hold.

First, we are going to state a compactness criterion for probability measures on $(\Omega, \mathcal{M})$. The proof of this criterion is very similar to the proof of Prokhorov's criterion of compactness in the case of Markov processes.

(1) Theorem : Let $\{A_k ; k \in Z^d\}$ be a set of positive numbers and denote by $\mathcal{P}$ the set of all probability measures P on $(\Omega, \mathcal{M})$ such that there is an $\eta \in E$ and a continuous function $c : E \to ([0, \infty))^{Z^d}$ satisfying $\|c_k\| \le A_k$, $k \in Z^d$, for which $P \sim \sum_k c_k\Delta_k$ starting from η. Then $\mathcal{P}$ is pre-compact (in the weak topology).

(2) Theorem : Let $\{c^{(n)}\}_1^\infty$ be a sequence of continuous function $c : E \to ([0, \infty))^{Z^d}$ such that $\|c_k^{(n)} - c_k\| \to 0$ as $n \to \infty$ for each $k \in Z^d$. Finally, for each n, let $P^{(n)} \sim \sum_k c_k^{(n)} \Delta_k$ starting from $\eta^{(n)}$ and assume that $\eta^{(n)} \to \eta$. Then $\{P^{(n)} : n \geq 1\}$ is pre-compact and any limit solves the martingale problem for $\sum_k c_k \Delta_k$ starting from η.

Proof : The compactness assertion is immediate from Theorem (1). Next, suppose that $\{P^{(n')}\}$ is a subsequence of $\{P^{(n)}\}$ and that $P^{(n')} \to P$. We must show that $P \sim L \equiv \sum_k c_{k\,k}$ starting from η. Clearly $P(\eta(0) = \eta) = 1$, and therefore we need only check that for all $0 \leq t_1 \leq t_2$, all bounded continuous M_{t_1}-measurable $F : \Omega \to R^1$, and all $\phi \in \mathcal{D}$:

$$E^P[(\phi(\eta(t_2)) - \phi(\eta(t_1)))F] = E^P[(\int_{t_1}^{t_2} L\phi(\eta(s))ds)F]. \tag{3}$$

But (3) holds when P and L are replaced by $P^{(n)}$ and $L^{(n)} \equiv \sum_k c_k^{(n)} \Delta_k$, respectively. Thus, since $L^{(n)}\phi \to L\phi$ uniformly, (3) follows from elementary properties about weak convergence of measures on (Ω, M).

(4) Theorem : Let $c : E \to ([0, \infty))^{Z^d}$ be a continuous function and define $L = \sum_k c_k \Delta_k$. For each $\eta \in E$, let $S(\eta)$ be the set of all probability measures P on (Ω, M) such that $P \sim L$ starting from η. Then for each η the set $S(\eta)$ is a non-empty, compact, convex subset of the set of probability measures on (Ω, M). Moreover, there is a measurable mapping $\eta \to P_\eta \in S(\eta)$ such that $\{P_\eta : \eta \in E\}$ forms a homogeneous strong Markov family. Finally, if for some $\eta^0 \in E$ there is more than one element of $S(\eta^0)$, then there are at least two different measurable mappings $\eta \to P_\eta^{(1)} \in S(\eta)$ and $\eta \to P_\eta^{(2)} \in S(\eta)$

such that $\{P_\eta^{(i)} : \eta \in E\}$ is a homogeneous strong Markov family for $i = 1$ and 2.

Proof : That $S(\eta)$ is convex and compact is obvious (cf. Theorem (2) for compactness). To show that $S(\eta) \neq \emptyset$, first observe that there is hardly anything to do if one knows that $c_k \equiv 0$ for all but a finite number of k's. (Indeed, in this case L is bounded and so one can easily construct a Markov family having L as its generator. Since it is easy to check that if $\{P_\eta : \eta \in E\}$ is a Markov family with generator L, then $P_\eta \sim L$ starting at η, the proof of existence in this case is complete.) Next, for $N \geq 1$ define

$$c_k^{(N)} = \begin{cases} c_k & \text{if } |k| \leq N \\ \\ 0 & \text{if } |k| > N, \end{cases}$$

and set $L^{(N)} = \sum_k c_k^{(N)} \Delta_k$. We know that for any given η there is a $P^{(N)} \sim L^{(N)}$ starting from η. But, by Theorem (2), $\{P^{(N)} : N \geq 1\}$ is pre-compact and any limit point P solves the martingale problem for L starting from η. Thus existence has been established in general.

The possibility of choosing a map $\eta \to P_\eta \in S(\eta)$ with the desired properties, as well as the last part of the theorem, follows from the techniques of Krylov, Math. U. S. S. R. Izv., 7 # 3 (691-709) (1973) plus Theorem (2) above.

(5) Theorem : Let $c : E \to ([0, \infty))^{Z^d}$ be a continuous function and define $L = \sum_k c_k \Delta_k$. Assume that for each $\eta \in E$ there is exactly one $P_\eta \sim L$ starting from η. Then the family $\{P_\eta : \eta \in E\}$ is strongly Markov and Feller continuous. Moreover, if $\{c^{(n)}\}_1^\infty$ is a sequence of

continuous functions on $E \to ([0, \infty))^{Z^d}$ such that $\|c_k^{(n)} - c_k\| \to 0$ as $n \to \infty$ for each $k \in Z^d$ and if $P^{(n)} \sim \sum_k c_k^{(n)} \Delta_k$ starting from $\eta^{(n)}$, where $\eta^{(n)} \to \eta$, then $P^{(n)} \to P_\eta$.

Proof : All these assertions follow easily from the above.

(6) Remarks : It is clear from Theorem (5) that if the martingale problem for L is well-posed (ie. if for each η there is exactly one solution), then there is a Feller semi-group corresponding to L with the properties required for the development in section IV. On the other hand, in spite of Theorem (4) and its assertion about the existence of a Markov family associated with L, very little can be said without the knowledge that the martingale problem is well-posed.

VII. The Question of Uniqueness ; a Counterexample :

(1) Example : Let $d = 1$ and define $c_k(\cdot) \equiv 0$ if $k < 1$ and for $n \geq 0$, and for $2^n \leq k < 2^{n+1}$ define :

$$c_k(\eta) = \begin{cases} 1 & \text{if } \eta_k = -1 \text{ and } \exists \ell \in [2^{n+1}, 2^{n+2}), \eta_\ell = 1 \\ \\ 0 & \text{otherwise .} \end{cases}$$

Clearly one solution to the martingale problem with these coefficients starting at $\eta^{(0)}$ such that $\eta_k^{(0)} = -1$ for all k's is the probability measure P such that

$$P(\eta_k(\cdot) \equiv -1,\ k \in Z^d) = 1.$$

We will produce a second solution. Let

$$c_k^{(N)}(\cdot) = \begin{cases} c_k(\cdot) & \text{if } k \le 2^{N+1} \\ 0 & \text{if } k \le 2^{N+1} \end{cases}$$

and

$$\eta_k^{(N)} = \begin{cases} 1 & \text{if } k = 2^{N+1} \\ -1 & \text{otherwise.} \end{cases}$$

Let $P^{(N)} \sim L^{(N)}$ starting from $\eta^{(N)}$. For $0 \le n \le N$, define

$$\tau_n = \inf\{t \ge \tau_{n+1} : \exists k \in [2^n, 2^{n+1}) \quad \eta_k(t) = 1\},$$

where $\tau_{N+1} \equiv 0$. Note that

$$P^{(N)}(\tau_n - \tau_{n+1} > t \mid M_{\tau_{n+1}}) = e^{-2^n t} \quad \text{on} \quad \{\tau_{n+1} < \infty\}.$$

Thus if $t_n = \frac{n+1}{2^n}$, then

$$\begin{aligned} P^{(N)}(\tau_0 > \sum_{n=0}^{\infty} \frac{n+1}{2^n}) &\le P^{(N)}(\tau_0 > \sum_{n=0}^{N} \frac{n+1}{2^n}) \\ &\le P^{(N)}((0 \le \exists n \le N) \quad \tau_n - \tau_{n+1} > t_n \text{ and } \tau_{n+1} < \infty) \\ &\le \sum_{n=0}^{N} e^{-(n+1)} \le \sum_{n=0}^{\infty} e^{-(n+1)} = \frac{1}{e-1} < 1 . \end{aligned}$$

Now let P be any weak limit of $\{P^{(N)}\}_0^{\infty}$. Then $P \sim L$ starting from $\eta^{(0)}$ and

$$\begin{aligned} P(\eta_1(4) = -1) &\le \overline{\lim_{N\to\infty}}\, P^{(N)}(\eta_1(4) = -1) \\ &\le \overline{\lim_{N\to\infty}}\, P^{(N)}(\tau_0 > 4) \le \frac{1}{e-1} < 1. \end{aligned}$$

(2) Remarks : By improving on the preceding example, but still using

the same sort of ideas, one can produce an example of coefficients $c_k \in \mathcal{D}$ for which the martingale problem has more than one solution and yet the c_k's are uniformly positive and uniformly bounded (cf. Holley and Stroock, Ann. of Prob., vol. 4 # 2 (1976)). Of much greater interest is an example advertised by L. Gray and to appear in his thesis (Cornell Univ., 1977). In his example the c_k's are positive, bounded, continuous and "shift invariant" (i.e. $c_k = c_0 \circ S^k$, where S is the shift map on E). As we are about to see, there can be no such example with $c_k \in \mathcal{D}$.

VIII. Uniqueness Under Liggett's Condition :

In general, one of the better techniques for proving uniqueness is the following. Given $L = \sum_k c_k \Delta_k$ on $\mathcal{D}$, one finds a class $\mathcal{D} \subseteq D(L) \subseteq C(E)$ with the property that L can be extended to $D(L)$ in such a way that (i) for every $f \in D(L)$, $Lf \in C(E)$ and

$$(f(\eta(t)) - \int_0^t Lf(\eta(s))ds, M_t, P)$$

is a martingale for all solutions P to the martingale problem for L ; and (ii) there is a determining set of ϕ's in $C(E)$ for which the equation

$$\lambda f - Lf = \phi \tag{1}$$

admits a solution $f \in D(L)$ whenever λ is sufficiently large. If such a class $D(L)$ exists, then one can show that for all $P \sim L$ starting from η:

$$E^P[\int_0^\infty e^{-\lambda t} \phi(\eta(t))dt] = f(\eta) \tag{2}$$

where $f \in D(L)$ is the solution to (1). From (2) one knows that the 1-dimensional time marginals of P are unique ; and once this is known it is quite easy to combine the last part of Theorem (3) in section V with standard "Markovian" reasoning to obtain the uniqueness of P itself. We are now going to give an example of conditions on the c_k's for which the above technique works.

Let $C^1(E)$ stand for the class of functions $f \in C(E)$ such that

$$|||f||| \equiv \sum_k \|f,_k\| < \infty .$$

The following lemma is very easily proved.

(3) Lemma : Suppose that $L = \sum_k c_k \Delta_k$ on $\mathcal{D}$, where $\sup_k \|c_k\| < \infty$. Then L admits a unique extention to $C^1(E)$ given by :

$$Lf = \lim_{N \uparrow \infty} \sum_{|k| \le N} c_k f,_k$$

Moreover, $L : C^1(E) \to C(E)$, and if $P \sim L$, then for all $f \in C^1(E)$:

$$(f(\eta(t)) - \int_0^t Lf(\eta(s))ds, M_t, P)$$

is a martingale.

We will say that coefficients $c : E \to [0, \infty)^{Z^d}$ satisfy Liggett's condition if

$$\text{(L. C.)} \begin{cases} \text{i)} \quad c \text{ is continuous} \\ \\ \text{ii)} \quad \sup_k (\|c_k\| + |||c_k|||) < \infty. \end{cases}$$

Under Liggett's condition, we will now show that (1) admits a solution $f \in C^1(E)$ for all sufficiently large λ and all $\phi \in C^1(E)$. Given

$c : E \to [0, \infty)^{Z^d}$ satisfying (L. C.), define $c^{(n)}$ by :

$$c_k^{(n)} = \begin{cases} c_k & \text{if } |k| \le n \\ 0 & \text{if } |k| > n, \end{cases}$$

and let $L^{(n)} = \sum_k c_k^{(n)} \Delta_k$. Clearly $L^{(n)}$ is bounded and generates a unique Feller semi-group $\{T_t^{(n)} : t \ge 0\}$ on $C(E)$ given by $T_t^{(n)} = e^{tL^{(n)}}$. Thus $R_\lambda^{(n)} = \int_0^\infty e^{-\lambda t}\, T_t^{(n)}\, dt$ is well defined on $C(E)$ into $C(E)$ for all $\lambda > 0$, and $\|R_\lambda^{(n)} \phi\| \le \frac{1}{\lambda}\|\phi\|$.

<u>(4)</u> <u>Lemma</u> : Define

(5) $$\gamma = \inf_k \inf_\eta \left(c_k(\eta) + c_k({}^k\eta)\right) - \sup_k \sum_{\ell \ne k} \|c_{k,\ell}\|$$

Then for each $\lambda > \sup_\ell c_\ell$ and all $\phi \in C^1(E)$, there is an $f \in C^1(E)$ such that $\lambda f - Lf = \phi$. Moreover,

(6) $$\|f\| \le \frac{\|\phi\|}{\lambda + \gamma}$$

<u>Proof</u> : Let $f^{(n)} = R_\lambda^{(n)} \phi$. For any $k \in Z^d$,

$$\lambda f_{,k}^{(n)}\ (\eta) = \phi_{,k}\ (\eta) + \sum_\ell \left(c_\ell^{(n)}\ f_{,\ell}^{(n)}\right)_{,k}(\eta).$$

Since $f_{,k}^{(n)} \in C(E)$, E is compact, and $f_{,k}^{(n)}\ ({}^k\eta) = -f_{,k}^{(n)}(\eta)$, we can find $\eta^* \in E$ such that $f_{,k}^{(n)}(\eta^*) = \|f_{,k}^{(n)}\|$. Thus

$$\|\lambda f_{,k}^{(n)}\| = \phi_{,k}(\eta^*) + \sum_\ell \left(c_\ell^{(n)} f_{,\ell}^{(n)}\right)_{,k}(\eta^*).$$

For $\ell \ne k$, we write :

$$\begin{aligned}(c_\ell^{(n)} f_{,\ell}^{(n)})_{,k}(\eta^*) &= c_\ell^{(n)}({}^k\eta^*) f_{,\ell k}^{(n)}(\eta^*) + c_{\ell,k}^{(n)}(\eta^*) f_{,\ell}^{(n)}(\eta^*) \\ &\le \|c_{\ell,k}^{(n)}\|\ \|f_{,\ell}^{(n)}\|\end{aligned}$$

since $f^{(n)}_{,\ell k}(\eta^*) = f^{(n)}_{,k}({}^{\ell}\eta^*) - f^{(n)}_{,k}(\eta^*) \le 0$. For $\ell = k$, we note that :

$$\begin{aligned}(c^{(n)}_k f_{,k})_{,k}(\eta^*) &= c^{(n)}_k({}^k\eta^*)f^{(n)}_{,k}({}^k\eta^*) - c^{(n)}_k(\eta^*)f^{(n)}_{,k}(\eta^*) \\ &= -(c^{(n)}_k({}^k\eta^*) + c^{(n)}_k(\eta^*))\|f^{(n)}_{,k}\| \\ &\le -\inf_{\eta}(c^{(n)}_k({}^k\eta) + c^{(n)}_k(\eta))\|f^{(n)}_{,k}\|\end{aligned}$$

Thus :

(7) $\lambda\|f^{(n)}_{,k}\| \le \|\phi_{,k}\| + \sum_{\ell\neq k}\|c^{(n)}_{\ell,k}\|\|f^{(n)}_{,\ell}\| - \inf_{\eta}(c^{(n)}_k({}^k\eta) + c^{(n)}_k(\eta))\|f^{(n)}_{,k}\|$.

In particular,

(8) $$\lambda\|f^{(n)}_{,k}\| \le \|\phi_{,k}\| + \sum_{\ell}\|c^{(n)}_{\ell,k}\|\|f^{(n)}_{,\ell}\| .$$

Now define

$$\alpha^{(N)}_k = \sup_{n\le N}\|f^{(n)}_{,k}\| .$$

Then, from (8) :

$$\lambda\sum_k\alpha^{(N)}_k \le |||\phi||| + \sum_{|\ell|\le N}|||c_\ell|||\alpha^{(N)}_\ell$$

Thus $\sum_k\alpha^{(N)}_k < \infty$ and, in fact,

$$\sum_k\alpha^{(N)}_k \le \frac{|||\phi|||}{\lambda - \sup_\ell|||c_\ell|||} .$$

Since $\alpha^{(N)}_k \nearrow \sup_n\|f^{(n)}_{,k}\|$, we have now shown that

(9) $$\sum_k\sup_n\|f^{(n)}_{,k}\| \le \frac{|||\phi|||}{\lambda - \sup_\ell|||c_\ell|||} .$$

From (9), we can show that $\{f^{(n)}\}^{\infty}_1$ is pre-compact in $C(E)$. Indeed, given $\varepsilon > 0$, choose $L \ge 1$ so that

$$\sum_{|k|>L}\sup_n\|f^{(n)}_{,k}\| < \varepsilon$$

Let η and η' be elements of E such that $\eta_k = \eta'_k$, $|k| \le L$. Then it is easy to see that for any $\psi \in C(E)$:

$$|\psi(\eta) - \psi(\eta')| \le \sum_{|k|>L} \|\psi,_k\| .$$

In particular, for any m,

$$\|f^{(m)}(\eta) - f^{(m)}(\eta')\| \le \sum_{|k|>L} \sup_n \|f^{(n)}_{,k}\| < \varepsilon$$

Since $\sup_m \|f^{(m)}\| \le \frac{\|\phi\|}{\lambda}$, we now know that $\{f^{(m)}\}_1^\infty$ is pre-compact in $C(E)$. Let $\{f^{(n')}\}_1^\infty$ be a convergent subsequence of $\{f^{(m)}\}_1^\infty$, and let f be its limit. Using (9) and the Lebesgue dominated convergence theorem, one can easily see that $f \in C^1(E)$ and that

$$\lambda f - Lf = \phi .$$

Finally, we want to show that (6) holds. But letting $n \to \infty$ in (7), we see that

$$\lambda\|f,_k\| \le \|\phi,_k\| + \sum_{\ell \ne k} \|c_{\ell,k}\| \, \|f,_\ell\| - \inf_\eta (c_k({}^k\eta) + c_k(\eta))\|f,_k\| .$$

Summing over k, we obtain :

$$\lambda|||f||| \le |||\phi||| + \sup_\ell |||c_\ell||| \, |||f||| - \inf_k \inf_\eta (c_k({}^k\eta) + c_k(\eta))|||f||| .$$

Since we already know that $|||f||| < \infty$, this implies (6).

<u>(10)</u> <u>Corollary</u> : Let $c : E \to [0, \infty)^{Z^d}$ satisfy (L, C) and define $L = \sum_k c_k \Delta_k$ on $\mathcal{D}$. Then for each $\eta \in E$ there is exactly one solution P_η to the martingale problem for L starting from η. Moreover, the set $\{P_\eta : \eta \in E\}$ form a Feller continuous strongly Markovian family. Finally, if $\{T_t : t > 0\}$ is the Feller semi-group determined by

$\{P_\eta : \eta \in E\}$, then T_t maps $C^1(E)$ into itself; and in fact,

(11) $$|||T_t\phi||| \le e^{-\gamma t}\,|||\phi|||,$$

where γ is the number defined in (5).

Proof : The proof of everything except (11) is accomplished in the way outlined at the beginning of this section. (See Stroock and Varadhan, Comm. Pure and Appl. Math. 22 (1969) for more details.)

To prove (11), let $R_\lambda = \int_0^\infty e^{-\lambda t}\, T_t\, dt$, $\lambda > 0$. Then it is easy to identify $R_\lambda\phi$, $\lambda > \sup_\ell |||c_\ell|||$ and $\phi \in C^1(E)$, as the function f in Lemma (4) above. (Simply observe that

$$(e^{-t\lambda} f(\eta(t)) + \int_0^t e^{-\lambda s}\, \phi(\eta(s))ds,\ \mathcal{M}_t,\ P_\eta) \text{ is a martingale.)}$$

Thus, by (6)

$$|||R_\lambda\phi||| \le \frac{|||\phi|||}{\lambda+\gamma}$$

But it is well-known that

$$T_t\phi = \lim_{\lambda\uparrow\infty} e^{-\lambda t} \sum_{n=0}^\infty (\lambda^2 t R_\lambda)^n \phi/n!,$$

and so (11) now follows easily.

(12) Corollary (Dobrushin and Sullivan) : Suppose that $c : E \to [o, \infty)^{Z^d}$ satisfies (L. C.) and assume that the number γ in (5) is positive. Let $\{T_t : T \ge 0\}$ be the Feller semi-group described in Cor (10). Then there is a unique stationary probability measure μ for $\{T_t : t \ge 0\}$. Moreover, for $\phi \in C^1(E)$:

(13) $$\| T_t\phi - \int \phi d\mu \| \le \frac{M}{\gamma} e^{-\gamma t}\, |||\phi|||,$$

where $M = \sup_k \|c_k\|$.

Proof : From the general theory of Feller semi-groups over a compact state space, we know that there is at least one stationary measure μ for $\{T_t : t > 0\}$. Moreover, it is obvious that if $\phi \in C(E)$ has the property that $T_t\phi$ converges to a constant as $t \to \infty$, then that constant must be $\int \phi d\mu$. Thus, it suffices for us to show that for each $\phi \in C^1(E)$ there is a constant A_ϕ such that $\|T_t\phi - A_\phi\| \le \frac{M}{\gamma} e^{-\gamma t} |\!|\!| \phi |\!|\!|$. But for any $f \in C^1(E)$, $\|Lf\| \le M |\!|\!| f |\!|\!|$. Thus, if $\phi \in C^1(E)$, then

$$|(T_t\phi - T_s\phi)| = \left|\int_s^t LT_u\phi \ du\right| \le M \int_s^\infty e^{-\gamma u} |\!|\!|\phi|\!|\!| \ du$$
$$= \frac{M}{\gamma} e^{-\gamma s} |\!|\!|\phi|\!|\!| .$$

Thus, $T_t\phi$ converges in $C(E)$ to some function $\psi_\phi \in C(E)$. Moreover

$$|\!|\!|\psi|\!|\!| \le \lim_{t\to\infty} |\!|\!| T_t\phi |\!|\!| = 0,$$

and therefore ψ_ϕ is some constant A_ϕ. Finally,

$$\|A_\phi - T_s\phi\| = \lim_{t\to\infty} \|T_t\phi - T_s\phi\| \le \frac{M}{\gamma} e^{-\gamma s} |\!|\!|\phi|\!|\!| .$$

(14) Example : One choice of rates c_k for the classical stochastic Ising model of Glauber is

$$c_k(\eta) = [1 + \exp(2\beta\eta_k \sum_{|\ell-k|=1} \eta_\ell)]^{-1}.$$

If $d = 1$, then we can write

$$c_k(\eta) = \frac{1}{2} + \frac{1}{4}[(1 + e^{4\beta})^{-1} - (1 + e^{-4\beta})^{-1}]\eta_k(\eta_{k-1} + \eta_{k+1}).$$

It is easy to see that

$$c_k({}^k\eta) + c_k(\eta) = 1$$

and

$$\sum_{\ell \neq k} \| c_{k,\ell} \| = \left| \frac{\sinh 4\beta}{1 + \cosh 4\beta} \right| < 1.$$

Thus Corollary (12) applies and shows both that there is only one Gibbs state for the Ising model in 1-dimension and that the standard Glauber model converges exponentially fast to that Gibbs state.

If $d = 2$, matters are less satisfactory. It is known that exactly one Gibbs state exists if and only if $\beta \leq \operatorname{arcsinh} 1$. Corollary (12) only tells us that the corresponding Glauber model converges to that Gibbs state when $\beta < (\log 3)/4$. On the other hand, for $\beta < (\log 3)/4$, it does yield exponentially fast convergence.

By entirely different arguments, Holley (Rocky Mount. J. of Math., 4(1974)) has shown that the standard stochastic Ising model will converge to the unique Gibbs measure whenever the Gibbs measure is unique. His result is dimension independent, but gives no rate of convergence.

(15) Remark : In section 5 of Holley and Stroock (Ann. of Prob., vol 4#2 (1976)) a quite different proof of the uniqueness part of Corollary (10) above is given. To facilitate the presentation of the idea behind that proof, assume $c_k > 0$ for all k's. It is shown that if $P \sim L = \sum_k c_k \Delta_k$ starting from η, then there is a right-continuous function $\xi : [0, \infty) \times \Omega \to E$ having left-limits and a continuous function $\sigma : [0, \infty) \times \Omega \to [0, \infty)^{Z^d}$ such that

i) the distribution of ξ under P solves the martingale problem for $\sum_k \Delta_k$ starting from η,

ii) $\sigma_k(t) = \int_0^t c_k(\xi \circ \sigma(u))du$, $t \geq 0$ and $k \in Z^d$, where
$(\xi \circ \sigma)_k = \xi_k \circ \sigma_k$,

iii) $\eta(\cdot) = \xi \circ \sigma(\cdot)$

The existence of $\xi(\cdot)$ and $\sigma(\cdot)$ have nothing to do with the condition (L. C.), they exist in general. Where (L. C.) plays and important role is in the proof that if

(16) $$\sigma_k(t) = \int_0^t c_k(\xi(\sigma(u)))du, \qquad k \in Z^d,$$

then $\sigma(t)$ must be $\xi(\cdot)$-measurable. Without this last fact, there is no way of using i), ii) and iii) to prove the uniqueness of P. The idea used to prove that any solution to (16) must be $\xi(\cdot)$-measurable is reminiscent of the technique used in Itô's approach to stochastic integral equations with Lipschitz continuous coefficients.

(17) Remark : Suppose that the c_k's come from a finite range potential J. Let $\{T_t : t > 0\}$ be the Feller semi-group determined by $L = \sum_k c_k\Delta_k$. Then any Gibbs state μ with potential J is a stationary measure for $\{T_t : t > 0\}$. Now suppose that $\{T_t : t > 0\}$ is ergodic (i.e. has only one stationary measure), and in addition, satisfies

(18) $$\left\| T_t\phi - \int \phi \, d\mu \right\| \leq A(\phi)e^{-\alpha t}, \qquad \phi \in \mathcal{D},$$

where $A(\phi)$ is some constant depending only on ϕ and $\alpha > 0$ is independent of ϕ. One can then show that μ satisfies

$$\left\| \int \phi \, d\mu - E^{\mu}[\phi \mid \mathcal{B}^{\Lambda_N}] \right\| \leq B(\phi)e^{-\beta(\phi)N}$$

where $\Lambda_N = \{k \in E : |k| \le N\}$, $B(\phi)$ is a constant depending only on ϕ, and $\beta(\phi) > 0$ depends only on the size of $\{k \in Z^d : \phi_{,k} \not\equiv 0\}$. For results of this sort, see Holley and Stroock: Comm. Math. Phys. 48(1976) and Z. Wahr, 35(1976).

Of course, (18) certainly implies that

$$(19) \qquad \left\| T_t\phi - \int \phi \, d\mu \right\|_{L^2(\mu)} \le e^{-\alpha t} \|\phi\|_{L^2(\mu)}, \qquad t \ge 0,$$

because (in the notation of section II) $E^\mu_\lambda - E^\mu_0 = 0$ for $\lambda < \alpha$. It would be very interesting to know if the existence of a $\mu \in G(L)$ and an $\alpha > 0$ for which (19) holds implies, in general, that $G(L) = \{\mu\}$; or, even better, that $\{T_t : t > 0\}$ is ergodic (i.e. has only one stationary measure).

IX. A Perturbation Technique :

Let $c : E \to [0, \infty)^{Z^d}$ be a continuous function and define L on $\mathcal{D}$ accordingly. As we have seen in section VIII, uniqueness of solutions to the martingale problem for L can be proved by constructing a version of a resolvent $(\lambda I - L)^{-1}$ in such a way that $\{(\lambda I - L)^{-1}\phi \in D(L)$ for large λ's and $\phi\}$ is a determining subset of $C(E)$. When dealing with various partial differential operators, one of the more successful approaches to studying resolvents has been through the use of perturbation theory. Unfortunately, perturbation techniques appear to be less well suited to the present situation ; although, as we are about to see, they yeild very good results when they do work.

Let λ denote the Haar measure on E. That is

$$\lambda = ((\delta_{\{-1\}} + \delta_{\{1\}})/2)^{Z^d}.$$

Given $c : E \to [0, \infty)^{Z^d}$, define

$$\overline{c}_k = \int c_k(\eta)\lambda(d\eta), \tag{1}$$

and let $\overline{L} = \sum_k \overline{c}_k \Delta_k$ be defined on $\mathcal{D}$. We are going to compute $\overline{R}_\lambda = (\lambda I - \overline{L})^{-1}$ in terms of its "Fourier transform". For this purpose, recall that $\hat{E} \equiv \{F \subseteq Z^d ; |F| < \infty\}$ can be identified with the character group of E via the map $F \to \chi_F$, where

$$\chi_F(\eta) = \prod_{k \in F} \eta_k. \tag{2}$$

The group operation on $\hat{E}$ is that of symmetric difference. That is, the "product" of F and G is

$$F \Delta G \equiv \{k ; k \in (F \cup G) \setminus (F \cap G)\}. \tag{3}$$

Obviously :

$$\chi_{F \Delta G} = \chi_F \cdot \chi_G. \tag{4}$$

Given $\phi \in L^2(\lambda)$, we define $\hat{\phi}$ on $\hat{E}$ by :

$$\hat{\phi}(F) = \int \phi(\eta)\chi_F(\eta)\lambda(d\eta). \tag{5}$$

The Fourier inversion formula in this context is

$$\phi = \sum_{F \in \hat{E}} \hat{\phi}(F) \chi_F, \tag{6}$$

where the convergence on the right hand side of (6) is in the sense of $L^2(\lambda)$ convergence of the partial sums :

$$\sum_{F \subseteq \Lambda_N} \hat{\phi}(F)\chi_F$$

where $\Lambda_N = \{k \;;\; |k| \leq N\}$. Next observe that for each $F \in \hat{E}$, χ_F is an eigenfunction of $\overline{L}$. In fact, since

$$\Delta_k \chi_F = \begin{cases} -2\,\chi_F & \text{if } k \in F \\ 0 & \text{if } k \notin F, \end{cases} \tag{7}$$

we have

$$\overline{L}\chi_F = -2\overline{c}_F \chi_F, \tag{8}$$

where

$$\overline{c}_F = \sum_{k \in F} \overline{c}_k. \tag{9}$$

Thus, if

$$\sigma(\overline{L}) = \overline{\{-2\overline{c}_F \;;\; F \in \hat{E}\}} \tag{10}$$

then for $\lambda \in C \backslash \sigma(\overline{L})$ we have

$$(\lambda I - \overline{L})\chi_F = (\lambda + 2\overline{c}_F)\chi_F,$$

and so,

$$\overline{R}_\lambda \chi_F = \frac{\chi_F}{\lambda + 2\overline{c}_F}\ . \tag{11}$$

It follows that for $\lambda \notin \sigma(\overline{L})$, $\overline{R}_\lambda$ is well defined as a bounded linear operator on $L^2(\lambda)$ into itself and that $\overline{R}_\lambda$ is determined by the equation:

$$\overline{R}_\lambda \phi(F) = \frac{\hat{\phi}(F)}{\lambda + 2\overline{c}_F}\,, \qquad F \in \hat{E}. \tag{12}$$

It is clear from the preceding that $\overline{R}_\lambda$ is not a compact operator on $L^2(\lambda)$ in most cases. For instance, if $\overline{c}_k = a > 0$ for all k, then $\overline{c}_F = a|F|$ and therefore $(\lambda + 2an)^{-1}$ is an eigenvalue of infinite multiplicity for each $n \geq 1$. This fact indicates that pertubation theory will not be very successful in $L^2(\lambda)$. For this reason, we are led to consider the Banach space $\hat{L}$ of function $\phi \in C(E)$ such that

(13) $$\|\phi\|_{\hat{L}} \equiv \sum_{F \in \hat{E}} |\hat{\phi}(F)| .$$

It is clear from (6) that

(14) $$\phi \leq \phi_{\hat{L}}.$$

Moreover, it follows from (12) that for $\lambda \notin \sigma(\mathcal{L})$ $\overline{R}_\lambda$ is bounded on $\hat{L}$ into itself. Finally, if

(15) $$a \equiv \inf_k \overline{c}_k > 0,$$

then for $\lambda \notin \sigma(\mathcal{L})$ one can easily see that $\overline{R}_\lambda$ maps $\hat{L}$ boundedly into $C^1(E)$. These remarks indicate that $\hat{L}$ is a good space in which to do perturbation theory.

We now assume that $c : E \to [0, \infty)^{Z^d}$ is a continuous function such that

(16) $$\sum_{F \neq \phi} |\hat{c}_k(F)| \leq \alpha \overline{c}_k, \qquad k \in Z^d,$$

where $0 \leq \alpha < 1$. Define $D(\mathcal{L})$ to be the set of $\phi \in \hat{L}$ such that

$$\sum_{F \in \hat{E}} \overline{c}_F |\hat{\phi}(F)| < \infty.$$

The next Lemma is easi y proved.

(17) Lemma : If $\phi \in D(\mathcal{\bar{L}})$, then both

$$\sum_k \bar{c}_k \phi_{,k} \qquad \text{and} \qquad \sum_k c_k \phi_{,k}$$

are absolutely convergent. Moreover,

$$\lim_{\Lambda \uparrow Z^d} \sum_{k \in \Lambda} \bar{c}_k \phi_{,k} \qquad \text{and} \qquad \lim_{\Lambda \uparrow Z^d} \sum_{k \in \Lambda} c_k \phi_{,k}$$

exist in $\hat{L}$. Finally, $\bar{R}_\lambda \hat{L} = D(\mathcal{\bar{L}})$.

Because of (12), we can extend the definition of $\mathcal{\bar{L}}$ and $\mathcal{L}$ to $D(\mathcal{\bar{L}})$. Also, it is clear that as an operator on $\hat{L}$ with domain $D(\mathcal{\bar{L}})$ $\mathcal{\bar{L}}$ is the generator of a strongly continuous contraction semi-group having resolvent operator $\bar{R}_\lambda$ defined by (12).

(18) Lemma : If $\lambda \notin \sigma(\mathcal{\bar{L}})$ and $\phi \in \hat{L}$, define

(19) $$A_\lambda \phi = (\mathcal{L} - \mathcal{\bar{L}})\bar{R}_\lambda \phi .$$

Then A_λ is bounded on $\hat{L}$ into itself and :

(20) $$\widehat{A_\lambda \phi}(F) = \sum_{G \neq F} \sum_{k \in G} \hat{c}_k (F \,\Delta\, G) \frac{-2}{\lambda + 2\bar{c}_G} \hat{\phi}(G) .$$

In particular,

(21) $$\|A_\lambda \phi\|_{\hat{L}} \leq \rho(\lambda) \|\phi\|_{\hat{L}} ,$$

where

(22) $$\rho(\lambda) = \alpha \sup_{F \neq \emptyset} \left| \frac{2\bar{c}_F}{\lambda + 2\bar{c}_F} \right| .$$

Proof : Let us prove (20) :

$$\widehat{\Delta_k \overline{R}_\lambda \phi}\,(G) = \begin{cases} 0 & \text{if } k \notin G \\ \dfrac{-2}{\lambda+2\overline{c}_G}\ \hat{\phi}(G) & \text{if } k \in G. \end{cases}$$

Thus, since $\widehat{c_k - \overline{c}_k}\,(H) = \begin{cases} 0 & \text{if } H = \emptyset \\ \hat{c}_k(H) & \text{if } H \neq \emptyset \end{cases}$,

$$\widehat{(c_k - \overline{c}_k)\Delta_k \overline{R}_\lambda \phi}(F) = \sum_{\substack{G\neq F \\ G\ni k}} \frac{-2c_k(F\Delta G)}{\lambda+2\overline{c}_G}\ \hat{\phi}\,(G),$$

and (20) clearly follows from this. To prove (21), note that :

$$\sum_F |\widehat{A_\lambda \phi}(F)| \le \sum_F \sum_{G\neq F} \sum_{k\in G} \frac{2|\hat{c}_k(F\Delta G)|}{|\lambda+2\overline{c}_G|}\,|\hat{\phi}\ (G)|$$

$$= \sum_G \Big(\sum_{k\in G} \sum_{F\neq G} |\hat{c}_k(F\Delta G)|\Big) \frac{2|\hat{\phi}(G)|}{|\lambda+2\overline{c}_G|}$$

$$\le \alpha \sum_G \Big(\sum_{k\in G} \overline{c}_k\Big) \frac{2|\hat{\phi}(G)|}{|\lambda+2\overline{c}_G|}$$

$$= \alpha \sum_G \frac{2\overline{c}_G}{|\lambda+2\overline{c}_G|}\,|\ \hat{\phi}(G)| \le \rho(\lambda)\,\|\phi\|_{\hat{L}}.$$

<u>(22)</u> <u>Lemma</u> : If $\lambda \notin \sigma(\overline{L})$ and $\rho(\lambda) < 1$, then

(23) $$B_\lambda \equiv (I - A_\lambda)^{-1} = \sum_{n=0}^{\infty} A_\lambda^n$$

exists as a bounded operator on $\hat{L}$ into itself and

(24) $$\|B_\lambda \phi\|_{\hat{L}} \le \frac{1}{1-\rho(\lambda)}\ \|\phi\|_{\hat{L}}.$$

Moreover, if

(25) $$R_\lambda \equiv \overline{R}_\lambda \circ B_\lambda,$$

then R_λ maps $\hat{L}$ onto $D(\bar{L})$ and

$$(\lambda I - L)R_\lambda\phi = \phi.$$

Finally, if $H = \{\lambda \in C\backslash\{0\} ; \lambda \in \sigma(\bar{L})$ and $\rho(\lambda) < 1\}$, then $\lambda \to R_\lambda\phi$ is analytic on H into $\hat{L}$.

(26) Theorem : Suppose $c : E \to [0, \infty)^{Z^d}$ is a continuous function which satisfies (16) for some $0 \le \alpha < 1$. Then the martingale problem for L is well-posed. Moreover, if $\{P_\eta ; \eta \in E\}$ is the set of solutions, then $\{P_\eta ; \eta \in E\}$ forms a Feller continuous, strong Markov family. Finally, if $\lambda > 0$, then

$$E^{P_\eta}\left[\int_0^\infty e^{-\lambda t}\phi(\eta(t))dt\right] = R_\lambda\phi(\eta), \qquad \phi \in \hat{L}, \tag{27}$$

where R_λ is defined by (25).

Proof : The theorem follows from the observation that if $\phi \in D(\bar{L})$ and $P \sim L$, then

$$\left(\phi(\eta(t)) - \int_0^t L\phi(\eta(s))ds, M_t, P\right)$$

is a martingale. Thus, if $\phi \in \hat{L}$ and $\lambda > 0$, then

$$\left(e^{-\lambda t} R_\lambda\phi(\eta(t)) + \int_0^t e^{-\lambda s}\phi(\eta(s))ds, M_t, P\right)$$

is a martingale.

(28) Lemma : Assume, in addition to (16), that $\bar{c}_k > 0$ for all $k \in Z^d$. Define A_0 on $\hat{L}$ by

$$\widehat{A_0\,\phi}(F) = -\sum_{G\neq F}\sum_{k\in G}\frac{\hat{c}_k(F\Delta G)}{\bar{c}_G}\hat{\phi}(G). \tag{29}$$

Then A_0 maps $\hat{L}$ into itself and

(30) $$\sup_{\lambda \geq 0} \|A_\lambda \phi\|_{\hat{L}} \leq \alpha \|\phi\|_{\hat{L}}.$$

Moreover, if $B_0 = \sum_0^\infty A_0^n$, then $\lim_{\lambda \downarrow 0} \|B_\lambda \phi - B_0 \phi\|_{\hat{L}} = 0$ for all $\phi \in \hat{L}$.

Finally, if $\pi : \hat{L} \to R^1$ is defined by

(32) $$\pi\phi = \widehat{B_0 \phi}\ (\emptyset), \quad \phi \in \hat{L},$$

then

(33) $$\lim_{\lambda \downarrow 0} \|\lambda\ R_\lambda \phi - \pi\phi\|_{\hat{L}} = 0, \quad \phi \in \hat{L}.$$

<u>Proof</u> : The estimate (30) is derived in the same way as (21). To prove (31), note first that

$$\widehat{A_\lambda \phi - A_0 \phi}\ (F) = \sum_{G \neq F} \sum_{k \in G} \hat{c}_k (F \Delta G) \frac{\lambda}{\bar{c}_G (\lambda + 2\bar{c}_G)} \hat{\phi}(G)$$

and so,

$$\|A_\lambda \phi - A_0 \phi\|_{\hat{L}} \leq \alpha \sum_{G \neq \emptyset} \frac{\lambda}{\lambda + \bar{c}_G} |\hat{\phi}(G)|.$$

Thus, by Lebesgue's dominated convergence theorem,

$$\|A_\lambda \phi - A_0 \phi\|_{\hat{L}} \to 0 \qquad \text{as} \quad \lambda \downarrow 0.$$

Together with (30), it is easy to go from this to (31). Finally, to prove (33), note that :

$$\begin{aligned} \lambda R_\lambda \phi - \pi\phi &= \lambda \bar{R}_\lambda \circ (B_\lambda \phi - \widehat{B_0 \phi}(\emptyset)) \\ &= \widehat{B_\lambda \phi}(\emptyset) - \widehat{B_0 \phi}\ (\emptyset) + \sum_{F \neq \emptyset} \frac{\lambda}{\lambda + 2\bar{c}_F} \widehat{B_\lambda \phi}(F) \end{aligned}$$

Thus

$$\begin{aligned}\|\lambda R_\lambda\phi - \pi\phi\|_{\hat{L}} &= |\widehat{B_\lambda\phi}(\emptyset) - \widehat{B_0\phi}(\emptyset)| + \sum_{F\neq\emptyset}\frac{\lambda}{\lambda+2\bar{c}_F}|\widehat{B_\lambda\phi}(F)| \\ &\le |\widehat{B_\lambda\phi}(\emptyset) - \widehat{B_0\phi}(\emptyset)| + \sum_{F\neq\emptyset}\frac{\lambda}{\lambda+2\bar{c}_F}|\widehat{B_0\phi}(F)| \\ &\quad + \sum_{F\neq\emptyset}\frac{\lambda}{\lambda+2\bar{c}_F}|\widehat{B_\lambda\phi}(F) - \widehat{B_0\phi}(F)| \\ &\le \|B_\lambda\phi - B_0\phi\|_{\hat{L}} + \sum_{F\neq\emptyset}\frac{\lambda}{\lambda+2\bar{c}_F}|\widehat{B_0\phi}(F)|.\end{aligned}$$

The first term tends to 0 by (31), and the second one by the dominated convergence theorem.

(34) Theorem : If $c : E \to [0, \infty)^{Z^d}$ is a continuous function satisfying (16) for some $0 \le \alpha < 1$ and if $\bar{c}_k > 0$ for all k, then the Feller semi-group $\{T_t \; ; \; t > 0\}$ determined by L has a unique stationary measure μ. That is, there is a probability measure μ on $(E, \mathcal{B})$ such that

$$\frac{1}{T}\int_0^T T_s\phi \, ds \to \int \phi \, d\mu$$

uniformly as $T \uparrow \infty$ for each $\phi \in C(E)$. Moreover, if $\phi \in \hat{L}$, then

(35) $$\int \phi d\mu = \pi\phi,$$

where π is defined in (32).

Proof : From the general theory of Feller semi-groups over compact space, we know that it is enough to check that there is at most one stationary measure for $\{T_t \; ; \; t > 0\}$. But if μ is stationary for $\{T_t \; ; \; t > 0\}$, then

$$\int \lambda R_\lambda \, \phi \, d\mu = \int \phi \, d\mu$$

for all $\lambda > 0$ and $\phi \in \hat{L}$. Hence, since $\lambda R_\lambda \phi \to \pi\phi$ as $\lambda \downarrow 0$, (35) holds for any stationary μ and all $\phi \in \hat{L}$. Clearly this proves that there is only one stationary μ.

(36) Remark : Suppose that, in addition to (16), the coefficient function c satisfies (15). One can then show that not only is the essential semi-group $\{T_t \,;\, t > 0\}$ ergodic, but also T_t converges to π strongly in $\hat{L}$ at an exponential rate (cf. section 7 of Holly & Stroock, Anal. of Prob., vol. 4#2(1976)). Since we are going to arrive at this fact by an entirely deferent approach in the next section, we will not derive it here.

Perhaps the most significant aspect of the preceding perturbation method is that it yields a reasonably explicit expression for the stationary measure in terms of the coefficients. In particular, one can use the Neumann series defining B_0 to prove analytic dependence of the stationary measure on the coefficients. See Holly & Stroock, Comm. Math. Phys., 48(1976) for more details.

(37) Remark : The fact that the χ_F's are characters of the group E plays no really essential role in the preceding development. What is important is that they are the eigenfunctions of the operator $\overline{L}$. With this observation in mind, consider the operator

$$L^0 = \sum_k \frac{a_k}{2} (1 + \alpha_k \eta_k) \, \Delta_k,$$

where $\{a_k \,;\, k \in Z^d\} \subseteq [0, \infty)$ and $\{\alpha_k \,;\, k \in Z^d\} \subseteq [0, 1]$. The eigenfunctions for this operator are the functions :

$$\chi_F^{(\alpha)}(\eta) = \prod_{k\in F} \frac{\alpha_k + \eta_k}{\alpha_k + 1}, \qquad F \in \hat{E};$$

and, in fact,

$$L^0 \chi_F^{(\alpha)} = -\left(\sum_{k\in F} a_k\right)\chi_F^{(\alpha)}.$$

We can now carry out the same program as the above for operators L of the form

$$L = \sum_{k\in Z^d} \frac{a_k}{2}\,(1 + \alpha_k\eta_k - (1 + \alpha_k)\eta_k \sum_{F\in\hat{E}} \gamma(k, F)\chi_F^{(\alpha)}(\eta))\Delta_k,$$

where

$$\sup_k \sum_{F\in\hat{E}} |\gamma(k, F)| < 1.$$

For more details, see Holly & Strook, Comm. Math. Phys., 48(1976).

(38) Remark : We cannot drop the assumption that α is strictly less than 1 in Theorem (34). For examples, let $c_k(\eta) = 1 + \eta_{k+1}$, $k \in Z^d$. Then it is easy to check that both $\lambda(\cdot)$ and the probability measure $\prod_{k\in Z^d} \delta_{\{-1\}}$ are stationary measures.

X. Dual Processes :

Suppose that $c : E \to [0, \infty)^{Z^d}$ is a continuous function with the property that

(1) $$\sum_{F\neq\emptyset} |\hat{c}_k(F)| \le \bar{c}_k, \qquad k \in Z^d.$$

Assume, for the moment, that in addition

(2) $$-\hat{c}_k(F) \ge 0, \quad k \in Z^d \text{ and } F \in \hat{E}\backslash\{\emptyset\}.$$

Given $F \in \hat{E}\backslash\{\emptyset\}$, we then have :

$$\begin{aligned}
L\chi_F(\eta) &= -2 \sum_{k\in F} c_k(\eta)\ \chi_F(\eta) \\
&= 2 \sum_{k\in F} \Big(\sum_{G\neq\emptyset} -\hat{c}_k\ (G)\chi_{F\Delta G}(\eta) - \bar{c}_k\chi_F(\eta)\Big) \\
&= 2 \sum_{k\in F} \sum_{G\neq\emptyset} -\hat{c}_k(G)(\chi_{F\Delta G}(\eta) - \chi_F(\eta)) \\
&\qquad - 2 \sum_{k\in F} (\bar{c}_k + \sum_{G\neq\emptyset} \hat{c}_k(G))\chi_F(\eta).
\end{aligned}$$

That is, if we define $\hat{L}$ on $C(\hat{E})$ by

(3) $$\hat{L}f(F) = 2 \sum_{k\in F} \sum_{G\neq\emptyset} -\hat{c}_k(G)(f(F \Delta G) - f(F)),$$

and if $V : \hat{E} \to [0, \infty)$ is given by

(4) $$V(F) = 2 \sum_{k\in F} (\bar{c}_k + \sum_{G\neq\emptyset} \hat{c}_k(G)),$$

then

(5) $$L\chi_F(\eta) = \hat{L}\ \chi_\eta(F) - V(F)\ \chi_\eta(F),$$

where $\chi_\eta : \hat{E} \to R^1$ is given by :

(6) $$\chi_\eta(F) = \chi_F(\eta).$$

Notice that, because of (2), $\hat{L}$ can be interpreted probabilistically as coming from a "Q-matrix" for a Markov chain on $\hat{E}$, and, because of (1), $V \geq 0$. Thus, if we assume that the Markov chain on $\hat{E}$ determined by $\hat{L}$ does <u>not explode</u> and if we denote by $\{\hat{P}_F ; F \in \hat{E}\}$ the associated family of probability measures on $D([0, \infty), \hat{E})$, then (5) would lead us to believe that :

(6) $$T_t \chi_F(\eta) = E^{\hat{P}_F}[\chi_\eta(F(t))e^{-\int_0^t V(F(s))ds}],$$

where $\{T_t : t > 0\}$ is any Markov semi-group on $B(E)$ satisfying :

(7) $$T_t\phi - \phi = \int_0^t T_s \mathcal{L}\phi \, ds, \qquad \phi \in \mathcal{D}.$$

Once one has (6), it is a simple matter to get :

(8) $$\widehat{T_t^*\mu}\,(F) = E^{\hat{P}_F}[\hat{\mu}(F(t))e^{-\int_0^t V(F(s))ds}]$$

for any probability measure μ on E simply by integrating both sides of (6) w. r. t. μ. In (8), T_t^* denotes the semi-group on probability measures given by

$$T_t^*\mu = \int P(t, \eta, \cdot)\mu(d\eta),$$

where $P(t, \eta, \cdot)$ denotes the transition probability function underlying $\{T_t ; t > 0\}$.

There are several uses to which (8) can be put. In the first place, it shows that there is at most one $\{T_t : t > 0\}$ satisfying (7). In conjunction with Theorem (4) of section VI, this proves uniqueness of solutions to the martingale problem for $\mathcal{L}$. Secondly, (8) is ideally suited to the study of ergodicity of $\{T_t : t > 0\}$. For example, if we define

$$\tau = \inf\{t \geq 0 ; F(t) = \emptyset\},$$

then, since $\emptyset$ is an absorbing state of the chain generated by $\hat{\mathcal{L}}$, we will have

$$\text{(9)} \qquad \lim_{t\uparrow\infty} \widehat{T_t^*\mu}\,(F) = E^{\hat{P}_F}\left[e^{-\int_0^\tau V(F(s))ds}\right], \qquad F \in \hat{E},$$

under the assumption that

$$\text{(10)} \qquad E^{\hat{P}_F}\left[e^{-\int_0^\infty V(F(s))ds}, \tau = \infty\right] = 0,$$

We turn now to a more rigorous development of these ideas. For each $N \geq 1$, let

$$\Lambda_N = \{k \in Z^d \; ; \; |k| \leq N\}.$$

Define $\hat{L}^{(N)}$ on $B(\hat{E})$ by

$$\text{(11)} \qquad \hat{L}^{(N)} f(F) = \begin{cases} \hat{L}f(F) & \text{if } F \subseteq \Lambda_N \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, for each $N \geq 1$, $\hat{L}^{(N)}$ generates a unique Markov family of probability measures $\{\hat{P}_F^{(N)} \; ; \; F \in \hat{E}\}$ on $\hat{\Omega} = D([0, \infty) \times \hat{E})$. Moreover, if

$$\zeta^{(N)} = \inf\{t \geq 0 \; ; \; F(t) \nsubseteq \Lambda_N\},$$

then for all $N \geq 1$ and $F \in \hat{E}$:

$$\text{(12)} \qquad \hat{P}_F^{(N+1)}\Big|_{\hat{M}_{\zeta^{(N)}}} = \hat{P}_F^{(N)}\Big|_{\hat{M}_{\zeta^{(N)}}}.$$

Thus, for each $F \in \hat{E}$ there is a unique $\hat{P}_F$ on $(\hat{\Omega}, \hat{M}_\zeta)$, where

$$\text{(13)} \qquad \zeta = \lim_{N\uparrow\infty} \zeta^{(N)},$$

such that

$$(14) \qquad \hat{P}_F\Big|_{\hat{M}_{\zeta^{(N)}}} = \hat{P}_F^{(N)}\Big|_{\hat{M}_{\zeta^{(N)}}}, \qquad N \geq 1.$$

Finally, if $f \in B(\hat{E})$, then

$$(f(F(t)) - \int_0^t \hat{L}^{(N)} f(F(s))ds,\ \hat{M}_t,\ \hat{P}_F^{(N)})$$

is a martingale, and therefore so is

$$(f(F(t \wedge \zeta^{(N)})) - \int_0^{t\wedge\zeta^{(N)}} \hat{L}f(F(s))ds,\ \hat{M}_{t\wedge\zeta^{(N)}},\ \hat{P}_F).$$

Combining these remarks with (5) and using the martingale version of the Feynman-Kac formula, we arrive at :

$$(15) \qquad T_t\chi_F(\eta) = E^{\hat{P}_F}[\chi_\eta(F(t))e^{-\int_0^t V(F(s))ds},\ \zeta^{(N)} > t]$$

$$+ E^{\hat{P}_F}[T_{t-\zeta^{(N)}}\ \chi_{F(\zeta^{(N)})}(\eta)e^{-\int_0^{\zeta^{(N)}} V(F(s))ds},\ \zeta^{(N)} \leq t]$$

for any Markov semi-group $\{T_t : t > 0\}$ satisfying (7). From (15) it is clear that we now have the next theorem.

(16) Theorem : If for each $F \in \hat{E}$:

$$(17) \qquad \hat{P}_F(\int_0^\zeta (1 + V(F(s)))ds < \infty) = 0,$$

then the martingale problem for L is well-posed. Moreover, in this case, the Feller semi-group $\{T_t : t > 0\}$ determined by L has the property that

$$(18) \qquad \widehat{T_t^*\mu}\,(F) = E^{\hat{P}_F}[\hat{\mu}(F(t))e^{-\int_0^t V(F(s))ds},\ \zeta > t].$$

(19) Lemma : If condition (1) is replaced by

(20) $$\sum_{F \neq \emptyset} |c_k(F)| \leq \alpha \bar{c}_k, \quad k \in Z^d,$$

for some $0 \leq \alpha < 1$, then (17) holds. If, in addition to (20),

(21) $$\bar{c}_k > 0, \quad k \in Z^d,$$

and if

(22) $$\tau = (\inf\{t \geq 0 \; ; \; F(t) = \emptyset\}) \wedge \zeta,$$

then

(23) $$\hat{P}_F\left(\int_0^\infty V(F(s))ds < \infty, \; \tau = \infty\right) = 0, \quad F \in \hat{E}.$$

Proof : Assume that (20) holds for some $0 \leq \alpha < 1$. Then it is easy to see that the operator $A^{(N)}$ given by :

$$A^{(N)} f(F) = \frac{1}{1+V(F)} \hat{L}^{(N)} f(F), \quad F \in \hat{E},$$

have the property that :

$$\|A^{(N)} f\| \leq \frac{1+\alpha}{1-\alpha} \|f\|, \quad f \in B(\hat{E}).$$

Thus, if $\tau(t)$ is defined by

$$\int_0^{\tau(t)} (1 + V(F(s)))ds = t, \quad t \geq 0,$$

then for all $T > 0$ and $F \in \hat{E}$:

$$\lim_{N\to\infty} \hat{P}_F^{(N)}(F(\tau(t)) \nsubseteq \Lambda_N \text{ for some } 0 \leq t \leq T) = 0,$$

since $F(\tau(t))$ is distributed under $\hat{P}_F^{(N)}$ in the same way as the Markov chain on $\hat{E}$ starting at F and having $A^{(N)}$ as its generator. Since

$$\int_0^{\zeta^{(N)}} (1 + V(F(s)))ds = \inf\{t \geq 0 \; ; \; F(\tau(t)) \notin \Lambda_N\},$$

this completes the proof of (17) from (20). The derivation of (23) from (20) plus (21) is similar.

Obviously, Theorem (16) together with Lemma (19) provides us with another proof of the first part of Theorem (26) in section IX, at least in the case when (2) obtains. What is more interesting is that it allows us to make a considerable improvement in our statement of Theorem (34) in section IX.

<u>(24)</u> <u>Theorem</u> : If (2), (20), and (21) hold and if $\{T_t : t > 0\}$ is the Feller semi-group determined by L, then $\{T_t : t > 0\}$ admits exactly one stationary measure μ and equation (35) of section IX holds for $\phi \in \hat{L}$. Moreover, if $\phi \in \hat{L}$, then

(25) $$\|T_t\phi - \pi\phi\|_{\hat{L}} \to 0 \qquad \text{as} \quad t \uparrow \infty,$$

and therefore

(26) $$\left\| T_t\phi - \int \phi d\mu \right\| \to 0 \qquad \text{as} \quad t \uparrow \infty$$

for all $\phi \in C(E)$. Finally, if in addition to (2) and (20) one has

(17) $$a = \inf_k \bar{c}_k > 0,$$

then for $\phi \in \hat{L}$:

(28) $$\|T_t\phi - \pi\phi\|_{\hat{L}} \leq 2e^{-\gamma t}\|\phi\|_{\hat{L}}, \qquad t > 0,$$

where

$$(29) \qquad \gamma = 2a(1 - \alpha).$$

In either case, one has

$$(30) \qquad \hat{\mu}(F) = E^{\hat{P}_F}\left[e^{-\int_0^\tau V(F(s))ds}\right], \quad F \in \hat{E},$$

where τ is defined by (22).

Proof : To prove (25) and (28), we start with the equation

$$(31) \qquad T_t\chi_F(\eta) = E^{\hat{P}_F}\left[\chi_\eta(F(t))e^{-\int_0^t V(F(s))ds}, \zeta > t\right].$$

Since $\emptyset$ is absorbing and $V(\emptyset) = 0$, the right hand side of (31) can be written as

$$E^{\hat{P}_F}\left[e^{-\int_0^\tau V(F(s))ds}, \tau \le t < \zeta\right] + E^{\hat{P}_F}\left[\chi_\eta(F(t))e^{-\int_0^t V(F(s))ds}, \tau > t\right]$$

$$= E^{\hat{P}_F}\left[e^{-\int_0^\tau V(F(s))ds}\right] + E^{\hat{P}_F}\left[\chi_\eta(F(t))e^{-\int_0^t V(F(s))ds}, \tau > t\right]$$

$$- E^{\hat{P}_F}\left[e^{-\int_0^\tau V(F(s))ds}, \tau > t\right].$$

(We have used here the fact that $\tau \le t < \zeta \Rightarrow \zeta = \infty$.) Thus

$$(32) \qquad \left\| T_t\chi_F - E^{\hat{P}_F}\left[e^{-\int_0^\tau V(F(s))ds}\right] \right\|$$

$$\le 2E^{\hat{P}_F}\left[e^{-\int_0^t V(F(s))ds}, \tau > t\right].$$

Assuming (21), we have from (23) and (32) :

$$\text{(33)} \qquad \lim_{t\uparrow\infty} \left\| T_t \chi_F - E^{\hat{P}_F}\left[e^{-\int_0^\tau V(F(s))ds}\right] \right\| = 0, \; F \in \hat{E}.$$

Clearly (33) identifies the stationary measure μ of $\{T_t : t > 0\}$ as the one satisfying (30). It is now obvious from (35) of section IX plus (33) above that (25) and therefore also (26) hold. Also, if (27) obtains, then $V(F) \geq 2a(1 - \alpha)$ for all $F \in \hat{E}$, and so (28) follows easily from (32).

<u>(34)</u> <u>Remark</u> : The assumption (2) is unnecessary and can be removed by the following trick. Let ∞ denote an abstract point which is not an element of Z^d. Define

$$\text{(35)} \qquad \mathcal{E} = (\{-1, 1\})^{Z^d \cup \{\infty\}} = E \times \{-1, 1\}$$

and

$$\text{(36)} \qquad \hat{\mathcal{E}} = \{\tilde{F} \subseteq Z^d \cup \{\infty\} \; ; \; |\tilde{F}| < \infty\}.$$

Given a continuous function $c : E \to ([0, \infty))^{Z^d}$ satisfying (1) (but not necessarily (2)), define γ on $Z^d \times (\hat{\mathcal{E}} \setminus \{\emptyset\})$ so that $\gamma(k, \{\infty\}) \equiv 0$ and for $F \in \hat{E} \setminus \{\emptyset\}$:

$$\gamma(k, F) = \begin{cases} -\hat{c}_k(F) & \text{if } \hat{c}_k(F) \leq 0 \\ 0 & \text{if } \hat{c}_k(F) > 0 \end{cases}$$

and

$$\gamma(k, F \cup \{\infty\}) = \begin{cases} 0 & \text{if } \hat{c}_k(F) \leq 0 \\ \hat{c}_k(F) & \text{if } \hat{c}_k(F) > 0. \end{cases}$$

Next, define $\tilde{c} : \mathcal{E} \to [0, \infty)^{Z^d \cup \{\infty\}}$ by

$$(37) \qquad \tilde{c}_k(\tilde{\eta}) = \begin{cases} \bar{c}_k - \sum\limits_{\tilde{F} \neq \emptyset} \gamma(k, \tilde{F}) \chi_{\tilde{F}}(\eta) & \text{if } k \in Z^d \\ \\ 0 & \text{if } k = \infty. \end{cases}$$

(We, of course, mean here that $\chi_{\tilde{F}}(\tilde{\eta}) = \prod\limits_{k \in \tilde{F}} \tilde{\eta}_k$, $\tilde{\eta} \in \mathcal{E}$ and $\tilde{F} \in \hat{\mathcal{E}}$.) If we now replace E by $\mathcal{E}$ and $\hat{E}$ by $\hat{\mathcal{E}}$, then it is clear that $\tilde{c}$ satisfies both (1) and (2). Moreover,

$$(38) \qquad c_k(\eta) = \tilde{c}_k((\eta, -1)), \quad k \in Z^d \text{ and } \eta \in E.$$

Thus, if $\tilde{\mathcal{D}} = \{\phi \in C(\mathcal{E}) ; \phi,_k \equiv 0$ for all but a finite number of $k \in Z^d \cup \{\infty\}\}$ and if $\tilde{\mathcal{L}} = \sum\limits_k \tilde{c}_k \Delta_k$ is defined on $\tilde{\mathcal{D}}$, then a probability measures $\tilde{P}$ on $\tilde{\Omega} = D([0, \infty), \mathcal{E}) = D([0, \infty) \times E)$ $D([0, \infty), \{-1, 1\})$ solves the martingale problem for $\tilde{\mathcal{L}}$ starting at $(\eta, -1)$ if and only if

$$(39) \qquad \tilde{P} = P \times \delta_{\{-1\}},$$

where $P \sim \mathcal{L}$ starting at η and $\delta_{\{-1\}}$ is the probability measure on $D([0, \infty), \{-1, 1\})$ such that

$$\delta_{\{-1\}}(\eta_\infty(t) = -1, \; t \geq 0) = 1.$$

With these remarks in mind, we see that it is possible to remove assumption (2) by simply moving all our considerations from E to $\mathcal{E}$. Thus, after the appropriate changes in their statements have been made, Theorem (16), Lemma (19), and Theorem (24) can all be extended to the case in which (2) does not hold. For more details and for further applications, see Holly & Stroock, Dual processes and their application to infinite interacting systems, to appear in Advances in Math.

(40) Remark : Just as in remark (37) of section IX, it should be pointed out that there is no reason to restrict oneself to the χ_F's; one can use equally well the $\chi_F^{(\alpha)}$'s introduced there. Of particular interest has been the case in which $\alpha_k = 1$, $k \in Z^d$. This case was studied extensively by Holley and Liggett, Ann. of Prob., 4, (1975) and Harris, Ann. of Prob., 2, (1974). In this connection, there is an interesting paper by Gray and Griffeath to appear in Ann. of Prob. Using the technique given above, only now with $\alpha_k = 1$, $k \in Z^d$, they have discovered an L for which the martingale problem is well-posed but it is not true that the closure of

$$\text{Graph } (L) = \{(\phi, L\phi) \; ; \; \phi \in \mathcal{D}\}$$

is dense. As far as I know, theirs is the first such example. It would be most interesting to know if such an example occurs in the context of diffusion theory. What such examples demonstrate is that, in general, studying Markov processes from the martingale problem point of view can give results that purely analytic semi-group considerations cannot yield.

XI. Free Energy Methods :

The method described below was introduced by Holley (Comm. Math. Phys. 23(1971)) to study certain questions about Glauber type models related to a given potential $J = \{J_F \; ; \; F \in \hat{E} \setminus \{\emptyset\}\}$.

Basically, the idea is to introduce on the space $M(E)$ of probability measures μ on $(E, \mathcal{B})$ a"Liapounov function" to study the "w-limit set" of an initial distribution μ under the flow on $M(E)$ induced by the Glauber type process. The presentation given here is adopted from the forthcoming article by Moulin Ollagnier and Pinchon to appear in Comm. Math. Phys.

Let $J = \{J_F \; ; \; F \in \hat{E} \setminus \{\emptyset\}\}$ be a given potential. Let $c : E \to ([0, \infty))^{Z^d}$ be a continuous function satisfying the detailed balance condition (3) of siction II. Assume, in addition, that c satisfies;

$$(1) \qquad 0 < \inf_k \inf_\eta c_k(\eta) \le \sup_k \sup_\eta c_k(\eta) < +\infty$$

and that

(2) the martingale problem for $L = \sum_k c_k \Delta_k$ is well-posed.

Let $\{T_t \; ; \; t > 0\}$ denote the Feller semi-group determined by L.

Throughout this section we will be dealing with the following set-up The sequence $\{\Lambda_n \; ; \; n \ge 1\}$ is a strictly increasing sequence of cubes centered at the origin in Z^d such that $\Lambda_n \uparrow Z^d$. For each $n \ge 1$, define

$$(3) \qquad U_n(\cdot) = \sum_{F \subseteq \Lambda_n} J_F \chi_F(\cdot)$$

and for $n \ge 1$ and $\mu \in M(E)$;

$$(4) \qquad F_n(\mu) = \int U_n(\eta)\mu(d\eta) + \sum_{F \subseteq \Lambda_n} \mu([F, \Lambda_n]) \log \mu([F, \Lambda_n])$$

where in (4), and below, we use the notation

$$(5) \quad [F, \Lambda] = \{\eta \in E \; ; \; \eta_k = 1 \text{ for } k \in F \text{ and } \eta_k = -1 \text{ for } k \in \Lambda \backslash F\}$$

whenever $F \subseteq \Lambda \in \hat{E}$. Finally, given $\mu \in M(E)$, let

(6) $$\mu_t = T_t^* \mu \ .$$

(7) Lemma ; If $\mu \in M(E)$ has the property that

(8) $$\mu([F, \Lambda_n]) > 0, \qquad F \subseteq \Lambda_n$$

then $\mathcal{F}_n(\mu_t)$ is continuously differentiable at $t = 0$ and

(9) $$\frac{d\,\mathcal{F}_n(\mu_t)}{dt}\bigg|_{t=0} = -\frac{1}{2}\sum_{k\in\Lambda_n}\sum_{F\subseteq\Lambda_n}(\Gamma_n(k, F) - \Gamma_n(k, F_k))\log\frac{\Gamma_n(k, F)}{\Gamma_n(k, F)}$$

$$+ \sum_{k\in\Lambda_n}\sum_{F\subseteq\Lambda_n}\Gamma_n(k, F)(V_n(k, F) + \log\frac{\Gamma_n(k, F)}{\mu([F, \Lambda_n])} - \log\frac{\Gamma_n(k, F_k)}{\mu([F_k, \Lambda_n])})$$

where $F_k \equiv F \,\Delta\, \{k\}$,

(10) $$\Gamma_n(k, F) = \int_{[F, \Lambda_n]} c_k(\eta)\mu(d\eta),$$

and

(11) $$V_n(k, F) = -2\sum_{\substack{G\subseteq\Lambda_n \\ k\in G}}(-1)^{|G\cap(\Lambda_n\setminus F)|}\, J_G$$

Proof ; First note that

$$\frac{d}{dt}\int U_n(\eta)\mu_t(d\eta)\bigg|_{t=0} = \int \mathcal{L}U_n(\eta)\mu(d\eta)$$

$$= \sum_{k\in\Lambda_n}\int c_k\Delta_k U_n\mu(d\eta) = \sum_{k\in\Lambda_n}\sum_{F\subseteq\Lambda_n}(U_n(F_k) - U_n(F))\Gamma_n(k, F)$$

where

$$U_n(F) = \sum_{G\subseteq\Lambda_n}(-1)^{|G\cap(\Lambda_n\setminus F)|}\, J_G.$$

Since

$$U_n(F_k) - U_n(F) = \sum_{G\in\Lambda_n} ((-1)^{|G\cap\{k\}|} - 1)(-1)^{|G\cap(\Lambda_n\setminus F)|} J_G$$

$$= -2 \sum_{\substack{G\in\Lambda_n \\ G\ni k}} (-1)^{|G\cap(\Lambda_n\setminus F)|} J_G = V_n(k, F),$$

we now have ;

$$\frac{d}{dt}\int U_n(\eta)\mu_t(d\eta)\Big|_{t=0} = \sum_{k\in\Lambda_n}\sum_{F\subseteq\Lambda_n} V_n(k, F)\Gamma_n(k, F). \tag{12}$$

Next, because of (8),

$$\frac{d}{dt}\sum_{F\subseteq\Lambda_n} \mu_t([F, \Lambda_n])\log \mu_t([F, \Lambda_n])\Big|_{t=0}$$

$$= \sum_{F\subseteq\Lambda_n} (1 + \log \mu([F, \Lambda_n])) \int L\chi_{[F, \Lambda_n]}(\eta)\ (d\eta)$$

$$= \sum_{k\in\Lambda_n}\sum_{F\subseteq\Lambda_n} \log \mu([F, \Lambda_n])(\Gamma_n(k, F_k) - \Gamma_n(k, F)),$$

since

$$\sum_{F\subseteq\Lambda_n}\int L\chi_{[F, \Lambda_n]}(\eta)\mu(d\eta) = \int L1(\eta)\mu(d\eta) = 0.$$

Thus ;

$$\frac{d}{dt}\sum_{F\subseteq\Lambda_n} \mu_t([F, \Lambda_n])\log \mu_t([F, \Lambda_n])\Big|_{t=0}$$

$$= \sum_{k\in\Lambda_n}\sum_{F\subseteq\Lambda_n} (\Gamma_n(k, F_k) - \Gamma_n(k, F))\log \mu([F, \Lambda_n])$$

$$= \sum_{k\in\Lambda_n}\sum_{F\subseteq\Lambda_n} \Gamma_n(k, F)\log\frac{\mu([F_k, \Lambda_n])}{\mu([F, \Lambda_n])}$$

$$= \sum_{k\in\Lambda_n}\sum_{F\subseteq\Lambda_n} \Gamma_n(k, F)\left(\log\frac{\Gamma_n(k, F)}{\mu([F, \Lambda_n])} - \log\frac{\Gamma_n(k, F_k)}{\mu([F_k, \Lambda_n])}\right)$$

$$- \sum_{k\in\Lambda_n}\sum_{F\subseteq\Lambda_n} \Gamma_n(k, F)\log\frac{\Gamma_n(k, F)}{\Gamma_n(k, F_k)}$$

$$= -\frac{1}{2}\sum_{k\in\Lambda_n}\sum_{F\subseteq\Lambda_n}(\Gamma_n(k, F) - \Gamma_n(k, F_k))\log\frac{\Gamma_n(k, F)}{\Gamma_n(k, F_k)}$$

$$+\sum_{k\in\Lambda_n}\sum_{F\subseteq\Lambda_n}\Gamma_n(k, F)(\log\frac{\Gamma_n(k, F)}{\mu([F, \Lambda_n])} - \log\frac{\Gamma_n(k, F_k)}{\mu([F_k, \Lambda_n])}).$$

combining this with (12), we arrive at (9).

<u>(13)</u> <u>Lemma</u> : If $t > 0$ and $\mu \in M(E)$, then for all $F \subseteq \Lambda \in \hat{E} \setminus \{\emptyset\}$

(14) $$\mu_t([F, \Lambda]) > 0.$$

<u>Proof</u> ; Choose $0 < a < b$ so that

(15) $$a \le c_k(\cdot) \le b, \qquad k \in Z^d.$$

For $k \in Z^d$, define

$$\phi_k^+(t, \eta) = \frac{a + (-a + (b+a)\frac{1+\eta_k}{2})e^{-(a+b)t}}{a + b}, \qquad t \ge 0,$$

and

$$\phi_k^-(t, \eta) = \phi_k^+(t, {}^k\eta).$$

Let

$$c_k^+(\eta) = \begin{cases} b & \text{if} \quad \eta_k = 1 \\ a & \text{if} \quad \eta_k = -1 \end{cases}$$

and

$$c_k^-(\eta) = c_k^+({}^k\eta).$$

Note that

$$(16)\qquad \frac{\partial \phi_k^+}{\partial t} = - c_k^+(\eta)\eta_k e^{-(a+b)t},$$

$$(17)\qquad \frac{\partial \phi_k^-}{\partial t} = c_k^-(\eta)\eta_k e^{-(a+b)t},$$

$$(18)\qquad \Delta_k \phi_k^+ = - \eta_k e^{-(a+b)t},$$

and

$$(19)\qquad \Delta_k \phi_k^- = \eta_k e^{-(a+b)t}.$$

Thus ;

$$(20)\qquad \frac{\partial \phi_k^\pm}{\partial t} = c_k^\pm \Delta_k \phi_k^\pm, \qquad t \geq 0,$$

and clearly

$$(21)\qquad \phi_k^\pm(0, \eta) = \begin{cases} 1 & \text{if} \quad \eta_k = \pm 1 \\ 0 & \text{if} \quad \eta_k = \mp 1. \end{cases}$$

Given a set $F \in \hat{E}$, let $\gamma = \{\gamma_k \ ; \ k \in Z^d\}$ be the element of E such that $\gamma_k = 1$ if and only if $k \in F$. For $G \in \hat{E}$, define

$$(22)\qquad \psi_G(t, \eta) = \prod_{k \in G} \phi_k^{\gamma_k}(t, \eta) \qquad (\equiv 1 \ \text{ if } \ G = \emptyset).$$

Then, by (21),

$$(23)\qquad \psi_\Lambda(0, \eta) = \chi_{[F, \Lambda]}(\eta),$$

and by (16) - (19),

$$(24)\qquad \frac{\partial \psi_\Lambda}{\partial t} = \sum_{k \in \Lambda} c_k^{\gamma_k} \Delta_k \psi_\Lambda, \qquad t > 0.$$

Thus, by (24) together with (18) and (19) ;

$$\begin{aligned} L\psi_\Lambda(t,\ \eta) &= \sum_{k\in\Lambda} c_k(\eta)\Delta_k\psi_\Lambda(t,\ \eta) \\ &= \frac{\partial\psi_\Lambda}{\partial t} + \sum_{k\in\Lambda} (c_k^{\gamma_k}(\eta) - c_k(\eta))\gamma_k\eta_k\psi_{\Lambda\setminus k}(t,\ \eta)e^{-(a+b)t}. \end{aligned}$$

Thus ;

$$(25) \qquad L\psi_\Lambda \geq \frac{\partial\psi_\Lambda}{\partial t}$$

since $\psi_G \geq 0$ for all G's and, by (15) :

$$(c_k^{\gamma_k}(\eta) - c_k(\eta))\gamma_k\eta_k \geq 0$$

Finally, let $T > 0$ be fixed. Then

$$(\psi_\Lambda(T - t \wedge T,\ \eta(t \wedge T)) - \int_0^{t\wedge T} (\frac{\partial}{\partial s} + L)\psi_\Lambda(T - s,\ \eta(s))ds,\ M_t,\ P_\eta)$$

is a martingale, where $P_\eta \sim L$ starting from η. Hence, by (23) and (25),

$$\begin{aligned} P_\eta(\eta(T) \in [F,\ \Lambda]) &= \psi_\Lambda(T,\ \eta) + E^{P_\eta}[\int_0^T (\frac{\partial}{\partial s} + L)\psi_\Lambda(T - s,\ \eta(s))ds] \\ &\geq \psi_\Lambda(T,\ \eta). \end{aligned}$$

Because $\psi_\Lambda(T,\ \eta) > 0$ for all $T > 0$ and $\eta \in E$, we now have ;

$$\mu_T([F,\ \Lambda]) \geq \int\psi_\Lambda(T,\ \eta)\mu(d\eta) > 0.$$

<u>(26)</u> <u>Theorem</u> : Given $\mu \in M(E)$, μ_t satisfies (8) for all $t > 0$ and $n \geq 1$; and, therefore $F_n(\mu_t)$ is continuously differentiable for all $t > 0$ and $n \geq 1$. In particular,

$$(27) \qquad F_n(\mu_{t_2}) - F_n(\mu_{t_1}) = \int_{t_1}^{t_2} F_n'(\mu_s)ds \qquad n \geq 1$$

and $0 < t_1 < t_2$, where $F_n'(\mu_s)$ is given by the right hand side of (9) with μ_s replacing μ in (10).

We now make an assumption which isn't entirely necessary but nonetheless simplifies matters. Namely, we are going to assume that there is an $L \geq 2$ such that for all $k \in Z^d$

$$(28) \qquad J_F = 0 \quad \text{if} \quad F \ni k \quad \text{but} \quad F \nsubseteq \{\ell \in Z^d \; ; \; |\ell - k| \leq L\}.$$

Under the assumption (28), it is obviously possible to find c_k's satisfying the detailed balance condition such that

$$(29) \qquad c_{k,\ell} \equiv 0 \quad \text{for all} \quad k \in Z^d \quad \text{and} \quad |\ell - k| > L,$$

and we will assume that such a choice of the c_k's has been made. Finally, we will from now on take

$$(30) \qquad \Lambda_n = \{k \in Z^d \; ; \; |k| \leq nL\}, \qquad n \geq 1.$$

We now have the following result.

<u>(31) Theorem</u> : Given $\mu \in M(E)$, define

$$(32) \qquad \mathcal{F}(\mu) = \overline{\lim_{n\to\infty}} \frac{1}{|\Lambda_n|} \mathcal{F}_n(\mu).$$

Then, for all $0 \leq t_1 < t_2$;

$$(33) \qquad \mathcal{F}(\mu_{t_2}) - \mathcal{F}(\mu_{t_1}) \leq \int_{t_1}^{t_2} \mathcal{F}'(\mu_s)ds,$$

where

$$(34) \qquad 2\mathcal{F}'(\mu_s) \equiv \overline{\lim_{n\to\infty}} \frac{-1}{|\Lambda_n|} \sum_{k\in\Lambda_n} \sum_{F\subseteq\Lambda_n} \{(\Gamma_n(s, k, F) - \Gamma_n(s, k, F_k)) \times \log \frac{\Gamma_n(s, k, F)}{\Gamma_n(s, k, F_k)} \}$$

with

$$(35) \qquad \Gamma_n(s,\ k,\ F) = \int_{[F,\ \Lambda_n]} c_k(\eta)\mu_s(d\eta).$$

In particular, $F(\mu_t)$ is a non-increasing function of $t > 0$.

<u>Proof</u> : First observe that, from (1) and (9), there is a constant $B < \infty$ such that

$$(36) \qquad F_n'(\mu) \le B|\Lambda_n|$$

for all $n \ge 1$ and $\mu \in M(E)$ satisfying (8). Thus, since $F_n(\mu_t)$ is continuous at $t = 0$ even if (8) does not obtain, (27) continues to hold even at $t = 0$. Hence, by Fatou's Lemma, all that we need to do is to prove that

$$(37) \quad \overline{\lim_{n\to\infty}} \frac{2}{|\Lambda_n|} F_n'(\mu_s) \le \overline{\lim_{n\to\infty}} \frac{-1}{|\Lambda_n|} \sum_{k\in\Lambda_n} \sum_{F\subseteq\Lambda_n} \{(\Gamma_n(s,\ k,\ F) - \Gamma_n(s,\ k,\ F_k)) \times \log \frac{\Gamma_n(s,\ k,\ F)}{\Gamma_n(s,\ k,\ F_k)}\}$$

for all $s > 0$. But, from the detailed balance condition plus (29), it is easy to see that

$$V_n(k,\ F) = -\log \frac{\Gamma_n(k,\ F_k)}{\mu([F,\ \Lambda_n])} + \log \frac{\Gamma_n(k,\ F_k)}{\mu([F_k,\ \Lambda_n])}$$

for all $F \subseteq \Lambda_n$ and $k \in \Lambda_{n-1}$. Thus the second term on the right hand side of (9) is bounded in absolute value by some constant times $|\Lambda_n \backslash \Lambda_{n-1}|$. Since $\frac{|\Lambda_n\backslash\Lambda_{n-1}|}{|\Lambda_n|} \to 0$ as $n \to \infty$, this completes the proof.

<u>(38) Lemma</u> : Let $\emptyset \ne \Lambda \subseteq \Lambda' \in \hat{E}$ and suppose that $\phi \in C(E)$ is non-negative. Given some $\mu \in \hat{E}$, define

$$\Gamma(F) = \int_{[F,\ \Lambda]} \phi(\eta)\mu(d\eta), \qquad F \subseteq \Lambda,$$

and

$$\Gamma'(F) = \int_{[F,\ \Lambda']} \phi(\eta)\mu(d\eta), \qquad F \subseteq \Lambda'.$$

Assuming that $\Gamma'(F) \neq 0$ for any $F \subseteq \Lambda'$, we have for each $G \subseteq \Lambda$,

$$\sum_{\substack{F\subseteq\Lambda' \\ F\cap\Lambda=G}} (\Gamma'(F) - \Gamma'(F_k))\log\frac{\Gamma'(F)}{\Gamma'(F_k)} \tag{39}$$

$$\geq (\Gamma(G) - \Gamma(G_k))\log\frac{\Gamma(G)}{\Gamma(G_k)}$$

<u>Proof</u> : Define $\Psi : (0, \infty) \times (0, \infty) \to [0, \infty)$ by

$$\Psi(x, y) = (x - y)\log\frac{x}{y}.$$

Then Ψ is convex and homogeneous of degree 1. Thus, by Jensen's inequality, for any sequences $\{a_i\}_1^n$ and $\{b_j\}_1^n$ of positive numbers ;

$$\sum_{j=1}^{n} \Psi(a_j, b_j) \geq \Psi\left(\sum_{j=1}^{n} a_j, \sum_{j=1}^{n} b_j\right).$$

In particular, for any $G \subseteq \Lambda$;

$$\sum_{\substack{F\subseteq\Lambda' \\ F\cap\Lambda=G}} \Psi(\Gamma'(F), \Gamma'(F_k)) \geq \Psi(\Gamma(G), \Gamma(G_k)).$$

<u>(40)</u> <u>Definition</u> : Given $k \in Z^d$, let $S^k : E \to E$ be defined by :

$$(S^k\ \eta)_\ell = \eta_{k+\ell}, \qquad \ell \in Z^d$$

We will say that the coefficients $c : E \to ([0, \infty))^{Z^d}$ are <u>shift</u> <u>invariant</u> if $c_k = c_0 \circ S^k$ for all k.

(41) Lemma : If $c : E \to ([0,\infty))^{Z^d}$ is shift invariant and the martingale problem for $L = \sum_k c_k \Delta_k$ is well-posed, then the Feller semi-group $\{T_t : t \geq 0\}$ determined by L has the property that for all $\phi \in C(E)$:

$$(42) \qquad T_t(\phi \circ S^k) = (T_t\phi) \circ S^k \qquad t > 0 \qquad \text{and} \qquad k \in Z^d.$$

Proof : Note that $L(\phi \circ S^k) = (L\phi) \circ S^k$ for all $k \in Z^d$ and $\phi \in \mathcal{D}$. Thus, if $P_\eta \sim L$ starting from η and if $P_\eta^{(k)}$ is the distribution of $S^k\eta(\cdot)$ under P_η, then $P_\eta^{(k)} \sim L$ starting from S_η^k. Thus

$$\begin{aligned}(T_t(\phi \circ S^k))(\eta) &= E^{P_\eta^{(k)}}[\phi(\eta(t))] = E^{P_{S^k\eta}}[\phi(\eta(t))] \\ &= (T_t\phi)(S^k\eta).\end{aligned}$$

(43) Theorem : If, in addition to satisfying (1), (2), and (29), the c_k's are shift invariant, then for every shift invariant $\mu \in M(E)\backslash G(J)$ and all $t > 0$, we have :

$$(44) \qquad F(\mu_t) < F(\mu).$$

In particular, the only shift invariant $\{T_t : t > 0\}$ stationary measures are in $G(J)$.

Proof : First observe that, by (42), μ_t is shift invariant if μ is. Moreover, $G(J)$ is closed in $M(E)$; and, therefore, $\mu \notin G(J)$ implies $\mu_t \notin G(J)$ for all small enough $t > 0$. Thus, in view of Lemma (13), all that we need to do is to show that if $\mu \notin G(J)$ is shift invariant and satisfies (8) for all $n \geq 1$, then $F'(\mu) < 0$. To this end, observe that there is a $\Lambda \in \hat{E}\backslash\{\emptyset\}$ and an $A \subseteq \Lambda$ such that for some $\ell \in \Lambda$:

$$\int_{[A,\ \Lambda]} c\ (\eta)\mu(d\eta) \neq \int_{[A_\ell,\ \Lambda]} c_\ell(\eta)\mu(d\eta),$$

and so

$$a \equiv \left(\int_{[A,\ \Lambda]} c_\ell(\eta)\mu(d\eta) - \int_{[A_\ell,\ \Lambda]} c_\ell(\eta)\mu(d\eta)\right)$$

$$\times \log \frac{\int_{[A,\ \Lambda]} c_\ell(\eta)d\mu}{\int_{[A_\ell,\ \Lambda]} c_\ell(\eta)d\mu} > 0.$$

By shift invariance plus Lemma (38), this means that for any $n \geq 1$ and $k \in \Lambda_n$ for which there exists a $j \in Z^d$ such that

(45) $$S^j\Lambda \subseteq \Lambda_n \quad \text{and} \quad j + \ell = k,$$

we have

$$\sum_{\substack{F \subseteq \Lambda_n \\ F \cap S^j\Lambda = S^jA}} (\Gamma_n(k,\ F) - \Gamma_n(k,\ F_k))\log\frac{\Gamma_n(k,\ F)}{\Gamma_n(k,\ F_k)} \geq a.$$

But it is obvious that the ratio of the number of $k \in \Lambda_n$ for which there exists a $j \in Z^d$ such that (45) is satisfied to the total number of k's in Λ_n tends to one as $n \to \infty$. Thus, $\mathcal{F}'(\mu) \leq -a/2$.

XII In One and Two Dimensions, All Stationary Measures are Gibbsian :

Theorem (43) of section XI proves that, under reasonable conditions, all shift invariant, stationary measures of a Glauber-type model are Gibbs measures. Although this result is a step in the right direction, it is far from satisfactory. We are now going to show that, at least when

$d = 1$ or 2, one can remove the assumption of shift invariance. These results are taken from Holley & Stroock's paper to appear soon in Comm. Math. Phys..

The assumptions with which we will be working in this section are the following :

$c : E \to (0, \infty)^{Z^d}$ is chosen to satisfy the detailed balance condition (cf. (3) of section III) for some given potential $J = \{J_F \; ; \; F \in \hat{E}\setminus\{\emptyset\}\}$. In addition to being continuous, we will assume that (1) and (29) of section XI are satisfied by c. Of course, this means that Liggett's condition is satisfied and, therefore, that there is a unique Feller semi-group associated with $L = \sum_k c_k \Delta_k$. The notation in this section is the same as in the preceding.

(1) Lemma : If μ M(E) is stationary for $\{T_t \; ; \; t > 0\}$, then

$$\sum_{k\in\Lambda_n} \sum_{F\subseteq\Lambda_n} (\Gamma_n(k, F) - \Gamma_n(k, F_k)) \log \frac{\Gamma_n(k, F)}{\Gamma_n(k, F_k)} \tag{2}$$
$$\leq K \sum_{k\in\partial\Lambda_n} \sum_{F\subseteq\Lambda_n} |\Gamma_n(k, F) - \Gamma_n(k, F_k)|,$$

where $K < \infty$ is independent of μ and $n \geq 1$, and $\partial\Lambda_n \equiv \Lambda_n \setminus \Lambda_{n-1}$. (Here, and throughout, Λ_n is defined as in (30) of section XI.)

Proof : First note that, since $\mu = T_1^*\mu$, Lemma (13) of section XI garantees that (8) of section XI holds to all $n \geq 1$. Moreover, since $F_n(\mu_t) = F_n(\mu)$ for all $t > 0$, (9) of section XI allows us to write ;

(3) $$\sum_{k\in\Lambda_n}\sum_{F\subseteq\Lambda_n}(\Gamma_n(k, F) - \Gamma_n(k, F_k))\log\frac{\Gamma_n(k, F)}{\Gamma_n(k, F_k)}$$

$$= 2\sum_{k\in\partial\Lambda_n}\sum_{F\subseteq\Lambda_n}\Gamma_n(k, F)(V_n(k, F) + \log\frac{\Gamma_n(k, F)}{\mu([F, \Lambda_n])} - \log\frac{\Gamma_n(k, F_k)}{\mu([F_k, \Lambda_n])}),$$

since, as we already observed in the proof of Theorem (31) of section XI

$$V_n(k, F) = -\log\frac{\Gamma_n(k, F)}{\mu([F, \Lambda_n])} + \log\frac{\Gamma_n(k, F_k)}{\mu([F, \Lambda_n])}$$

for $k \in \partial\Lambda_{n-1}$ and $F \subseteq \Lambda_n$. Finally, by an easy change of variables, it is easy to see that the right hand side of (3) is equal to

$$\sum_{k\in\partial\Lambda_n}\sum_{F\subseteq\Lambda_n}(\Gamma_n(k, F) - \Gamma_n(k, F_k)) \times \{V_n(k, F) + \log\frac{\Gamma_n(k, F)}{\mu([F, \Lambda_n])} - \log\frac{\Gamma_n(k, F_k)}{\mu([F_k\ \Lambda_n])}\}.$$

Thus (2) holds with

$$K = 2\sup_k\sum_{F\ni k}|J_F| + \sup_k\log\frac{\sup_\eta c_k(\eta)}{\inf_\eta c_k(\eta)}\quad .$$

Before proceeding, we introduce the following notation ;

(4) $$\alpha_n(k) = \sum_{F\subseteq\Lambda_n}(\Gamma_n(k, F) - \Gamma_n(k, F_k))\log\frac{\Gamma_n(k, F)}{\Gamma_n(k, F_k)}, \qquad k \in \Lambda_n$$

and

(5) $$\beta_n(k) = \sum_{F\subseteq\Lambda_n}|\Gamma_n(k, F) - \Gamma_n(k, F_k)|, \qquad k \in \Lambda_n.$$

By Lemma (38) of section XI ;

(6) $\qquad \alpha_m(k) \le \alpha_n(k), \qquad 1 \le m \le n$ and $k \in \Lambda_m$.

<u>(7) Lemma</u> : For $n \ge 1$ and $k \in \Lambda_n$;

(8) $\qquad \beta_n(k) \le 2\ e\ C\ (\alpha_n(k))^{1/2}$;

where $C^2 = \sup_{k' \in Z^d} \|c_{k'}\|$

<u>Proof</u> : If $\alpha_n(k) = 0$, then so does $\beta_n(k)$. Since $\beta_n(k) \le 2 \sup_\ell \|c_\ell\|$, there is nothing to prove if $\alpha_n(k) \ge 2 \sup_\ell \|c_\ell\|$. If $0 < \alpha_n(k) < 2 \sup_\ell \|c_\ell\|$, If $0 < \alpha_n(k) < 2 \sup_\ell \|c_\ell\|$, proceed as follows. Given $0 < \varepsilon \le 2$, let $A_n^{(\varepsilon)} = \{F \subseteq \Lambda_n \ ; \ |\log (\Gamma_n(k, F)/\Gamma_n(k, F_k))| < \varepsilon\}$ and $B_n^{(\varepsilon)} = \{F \subseteq \Lambda_n \ ; \ F \notin A_n^{(\varepsilon)}\}$. Then

$$\beta_n(k) \le (e \ - 1) \sum_{F \in A_n^{(\varepsilon)}} \Gamma_n(k, F) + \frac{1}{\varepsilon} \alpha_n(k)$$

$$\le \varepsilon e^2 C^2 + \frac{1}{\varepsilon} \alpha_n(k).$$

In particular, we can take

$$\varepsilon^2 = \frac{\alpha_n(k)}{e^2\ C^2} \quad ,$$

and thereby complete the proof.

<u>(9) Lemma</u> : Define

$$\gamma_1 = \sum_{k \in \Lambda_1} \alpha_1(k)$$

and

$$\gamma_n = \sum_{k \in \Lambda_n} \alpha_n(k), \qquad n \ge 2$$

Then, for $N \geq 2$:

$$(10) \qquad \sum_1^N \gamma_n \leq 2\ eCK\ (2L)^{(d-1)/2}\ N^{(d-1)/2}\ \gamma_N^{1/2}$$

Proof : By (6) and (2) ;

$$\sum_1^N \gamma_n \leq \sum_{k\in\Lambda_N} \alpha_N(k) \leq K \sum_{k\in\partial\Lambda_N} \beta_N(k).$$

Thus, since $|\partial\Lambda_N| \leq (2LN)^{d-1}$, we have from (8) :

$$\sum_1^N \gamma_n \leq eCK \sum_{k\in\partial\Lambda_N} (\alpha_N(k))^{1/2} \leq eCK\ (2LN)^{(d-1)/2}\ \gamma_N^{1/2}.$$

(11) Theorem : If $d = 1$ or 2 and $\mu \in M(E)$ is stationary for $\{T_t,\ t > 0\}$, then $\mu \in \mathcal{G}(J)$

Proof : We first note that because the c_k's are positive it is easy to show that $\mu \in \mathcal{G}(J)$ if and only if $\Gamma_n(k, F) = \Gamma_n(k, F_k)$ for all $n \geq 1$ and $k \in \Lambda_n$. Thus, we need only show that $\gamma_n = 0$ for all $n \geq 1$. Set $A = eCK\ L^{(d-1)/2}$. Then by (10) :

$$\sum_1^N \gamma_n \leq A\ N^{(d-1)/2}\ \gamma_N^{1/2}, \qquad N \geq 2.$$

Suppose $\gamma_n \neq 0$ for some $n \geq 1$ and let n_0 be the smallest such n. Define $b_N = \sum_1^N \gamma_n$. Then, for $N > n_0$;

$$b_N b_{N-1} \leq b_N^2 \leq A^2 N^{d-1}(b_N - b_{N-1})$$

$$= A^2 N^{d-1}\ b_N b_{N-1}\ \left(\frac{1}{b_{N-1}} - \frac{1}{b_N}\right).$$

Thus,

$$\sum_{M=n_0+2}^N \left(\frac{1}{b_{M-1}} - \frac{1}{b_M}\right) \geq \frac{1}{A^2} \sum_{M=n_0+2}^N \frac{1}{M^{d-1}}$$

for all $N \geq n_0 + 2$. On the other hand

$$\sum_{M=n_0+2}^{N} \left(\frac{1}{b_{M-1}} - \frac{1}{b_M}\right) = \frac{1}{b_{n_0+1}} - \frac{1}{b_N} \leq \frac{1}{\gamma_{n_0}} ,$$

and so

$$\frac{1}{\gamma_{n_0}} \geq \frac{1}{A^2} \sum_{M=n_0+2}^{N} \frac{1}{M^{d-1}} , \qquad N \geq n_0 + 2.$$

If $d = 1$ or 2, this is impossible, since

$$\sum_{n_0+2}^{\infty} \frac{1}{M^{d-1}} = \infty.$$

XIII. Open Problems :

This field is very new and has too many open problems to make a complete list. I will wherefore point out only a few, most of which are already apparent in the preceding sections.

(1) In many ways the most disturbing aspect of this theory is that the link between the original problem of studying Gibbs states and the stochastic formulation is not very solid. To be precise, let $J = \{J_F \,;\, F \in \hat{E}\backslash\{\emptyset\}\}$ be a finite range potential (i.e. satisfy (29) of section XI for some L). Then there are infinitely many choices of $c : E \to ([0, \infty))^{Z^d}$ satisfying the detailed balance condition, all of which are positive and satisfy Liggett's condition. However, there appears to be no canonical choice among these c's. If one imposes

additional conditions (like shift invariance), the range of reasonable choices is narrowed to some extent, but there is still much ambiguity. Thus, the first problem is to find a sound mathematical way of choosing the c's.

(2) Assuming that there is no good answer to (1), the next problem is to find out whether all reasonable choices of c's, for a given J, predict the same results about $G(J)$. The motivation behind section IV was to provide a partial answer to this question. Those results say that the choice of c (so long as it is reasonable) is irrelevant so long as one is looking at the L^2-theory. Unfortunately, it is not enough to look at the L^2-theory for these semi-groups. Two obvious questions in this connection are the following

i) If for some $\mu \in G(J)$ one has the existence of an $\alpha > 0$ for which :

$$\left\| T_t^{\mu}\phi - \int \phi d\mu \right\|_{L^2(\mu)} \leq e^{-\alpha t} \|\phi\|_{L^2(\mu)}, \qquad \phi \in L^2(\mu),$$

does it follow that $G(J) = \{\mu\}$? Even better, can one show, under the above hypothesis, that $\{T_t : t > 0\}$ is ergodic?

ii) If $G(J)$ has exactly one element, is $\{T_t, t > 0\}$ ergodic? For $d = 1$ or 2, the results of section XII provides us with an affirmative answer. Is the restriction on dimension necessary?

(3) It seems reasonable to suppose that a statement about the "stability" of ergodicity ought to be possible. For example, is it true that whenever one has

$$\left\| T_t\phi - \int \phi d\mu \right\| \leq A_{\phi}\, e^{-\alpha t}, \qquad \phi \in \mathcal{D},$$

for some $\alpha > 0$, then the semi-group associated with coefficients "close" to those determining $\{T_t ; t > 0\}$ must also be ergodic (and converge at an exponential rate)? The most encouraging results in this direction are those in section IX ; but, of course, the assumptions made there are very special.

(4) For most applications, the Liggett condition is sufficient. Nonetheless, it would be interesting to understand better what are the origins of non-uniqueness for the martingale problem. In his thesis (Cornell, 1977), L. Gray has developed a technique for proving uniqueness in a wide range of cases. Any farther information on this subject would be of considerable interest.

(5) It has been conjectured by several people that when $k = 1$ and the coefficients c_k are reasonable (e.g. positive, shift invariant, and in $\mathcal{D}$), then the associated semi-group $\{T_t : t > 0\}$ should be ergodic. As a consequence of section XII plus F. Spitzer's theorem showing that $\mathcal{G}(J)$ has only one member when $k = 1$, we see that this conjecture is correct for c's satisfying the detailed balance condition relative to some J. On the other hand, if the c's are not associated with some potential, this conjecture seems to be very difficult to settle. For example, suppose that we restrict our attention to c's of the form :

$$c_k(\eta) = 1 + a\eta_k + b\eta_{k+1} + c\eta_k\eta_{k+1}, \qquad k \in Z,$$

where a, b, and c are chosen so that $c_k(\eta) > 0$ for all $\eta \in E$. (Apart from a multiplicative constant, every positive, shift invariant coefficients depending only on η_k and η_{k+1} has this form.) Using techniques of the sort developed in section X, Holley & Strrock ("Dual processes ... ") have verified the conjecture in all cases other than when a, b, c all have the same sign or when a and b have the same sign but c has the opposite sign. It should be mentioned in this connection that there are rumors of a counter-example to the general conjecture, but, as far as I know, none has as yet appeared in print.

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